ADAPTATION OF A FREE PRECESSION MAGNETOMETER TO MEASUREMENTS OF DECLINATION

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by

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PREFACE

The purpose of this paper is to describe the adaptation of a free nuclear precession magnetometer to the measurement of variations in declination as well as the magnitude of the earth's magnetic field.

The bias coil technique employed for these measurements offers a simple and accurate means for permanently recording such variations. Although equipment size was of secondary importance, results of this investigation indicate that a light weight declination variometer can easily be constructed.

The investigation and development of this new type of variometer was performed at the Varian Associates Research Laboratory, Palo Alto, California, during the period January to March, 1955, while the author was a student in the Engineering Electronics curriculum at the U. S. Naval Postgraduate School, Monterey, California.

The author wishes to thank Doctor Martin E. Packard, the Director of Nuclear Magnetic Resonance Spectroscopy Research at the laboratory, for his assistance and suggestions. The author is also indebted to Varian Associates for the opportunity of conducting this investigation and to Mr. Dolan Mansir for his help and cooperation. In conclusion, thanks are expressed particularly to Professor Carl E. Menneken of the U. S. Naval Postgraduate School for his encouragement and criticisms in the preparation of this paper.

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CHAPTER I

BACKGROUND AND HISTORY

Man has been studying the magnetic field of the earth for centuries, and measurements of this field have been made from deep in the earth's surface to several hundred miles into the atmosphere. The cause of this invisible phenomenon is still a mystery, but its effect has had profound influence on the course of history. Since the discovery of lodestone in the Eleventh Century, mariners have depended on this magnetic field for navigation on the sea. Today it is still relied upon aboard even the most modern ships.

The existence of this field of force must be observed by use of scientific instruments because, unlike many other natural phenomena, it can not be detected by the human senses. The instrument used to measure and record the value and perturbations of this field is called a magnetometer.

1. Components of the earth's magnetic field [6].

The earth's magnetic field is uniquely defined by the length and direction of the total intensity vector. In terrestrial magnetic research such as is conducted in observatories and land and marine surveys, this vector is generally determined completely. For ease and accuracy of measurement, the intensity and direction of the total vector are not measured directly, but instead suitable intensity and angular components are measured.

In recording observatories these components may be the three rectilinear intensities, X, Y, and Z. In land surveys it is most convenient

to measure D, I, and H (where X is the astronomic north, Y is the astronomic east, Z the vertical component, D the declination, I the inclination, and H the horizontal component in the magnetic meridian). For the practical requirements of magnetic prospecting such detailed analysis is not necessary. Generally, one component only is measured, and it is advantageous to select such a component as will bear the closest relationship to anomalies and position of geologic bodies.

2. Classes of magnetic instruments.

It is convenient to divide magnetic instruments and equipment into three broad classes, namely: 1) absolute instruments to determine the values of the magnetic elements; 2) variation instruments and equipment to determine periodic and irregular variations; 3) special purpose instruments such as the compass declinometer. According to application, magnetic instruments may be grouped into: 1) prospecting magnetometers; 2) magnetic theodolites; and 3) observatory instruments. Finally, according to principle of construction magnetic instruments can be divided into two groups, namely, instruments for determination of the direction of the total or partial field vectors, and instruments for the determination of the intensity of the field or its components. In the first group of instruments the direction of a component may be determined: 1) by observing the rest position of a magnet capable of rotation about a vertical axis or about a horizontal axis; 2) by determining the position for zero induction of a rotating coil; and 3) by calculating direction from intensity ratios.

More pertinent for consideration in this paper are the instruments for the determination of intensity components. Instruments of this type

may be divided into four groups. In the first, intensities are obtained by measurements of the period of oscillation of a magnet about its zero position; in the second, intensities may be derived from induction observations with rotating coils; in the third, by measuring the induction in soft iron bars; and in the fourth, by using a comparison of the magnetic component with some other known constant force.

In practice four kinds of comparison forces may be used: 1) magnetic fields from coils or magnets; 2) elastic forces; 3) gravity; and 4) the kinetic energy of electrons. It is possible to give a general theory of all types of magnetometers by setting up equations for the effect of the earth's magnetic field upon a magnetic needle moving freely in space and by combining these with a second set of equations involving whatever comparison forces are used to measure the magnetic effects.

The measurement of the vertical intensity of the earth's magnetic field is more difficult than the measurement of the horizontal intensity. The value of the vertical intensity is regularly deduced from measured values of horizontal intensity and inclination. However, since the inclination cannot be measured with very great accuracy, particularly when the value is close to the vertical, the reliability of the value of the horizontal and vertical intensities derived in this manner are limited. Present methods for obtaining great accuracy consist of using a null technique for balancing the effect of the earth's magnetic field by a known field and then computing the derived fields.

3. Modern magnetic instruments.

The modern declination variometer consists essentially of a small magnet attached to a light frame carrying a mirror, all of which is

suspended by a fine quartz fiber. Variations in the direction of the magnet are recorded photographically after amplification by means of an optical lever. In addition, light reflected from a fixed mirror is brought to focus on the magnetogram to form a reference base line. The variation in the perpendicular distance between this base line and the trace is the measure of the variation of the magnetic declination.

The modern horizontal intensity variometer operated in a similar manner except that the suspended magnet operates about an axis perpendicular to the magnetic meridian. Since the magnetic moment of a magnet varies with temperature, it is customary to provide some means of temperature compensation on intensity variometers by mechanical, magnetic, or optical means.

With the advent of permavar, a material of nearly constant permeability, it was possible to construct a simple vertical intensity variometer utilizing the principle of an induction variometer. This type of variometer has the advantage that only one component of the field is measured regardless of the changes in direction of the magnetic field.

In the field of magnetic prospecting the Schmidt balances and the Hotchkiss superdip are used extensively. The Schmidt vertical and horizontal balances have a magnetic system balanced on a knife-edge at right angles to the magnetic meridian. Its deflections are measured by means of an autocollimational telescope system. Field procedure in applying the magnetometer consists of taking three readings, interrupted by clamping and releasing, in the east position after the instrument has been carefully leveled and orientated. Then, the same number of readings are taken in the west position. These readings are averaged and a base

reading subtracted. To this the temperature correction, auxiliary magnet correction, and the base correction are added or subtracted, respectively.

The Hotchkiss superdip is an instrument intended essentially for the measurement of total intensity. A magnetic system is suspended on a horizontal steel axle on agate bearings in the magnetic meridian. Fastened to the steel blade is a counter arm whose angle with the magnetic axis may be varied. Attached to this counterarm is a small mass whose position on the arm determines the latitude adjustment. The angle which the arm makes with the magnetic axis controls primarily the sensitivity of the instrument. This instrument can also be used for the determination of inclination by removing the counter weight and assuming symmetrical mass disposition about the axis of rotation.

In addition to these precision instruments there are a great many other magnetic prospecting instruments which can be used in place of them, particularly when the great sensitivity which these instruments furnish is not required. The simplest representative of these instruments is the Swedish mining compass. This consists of a magnetic needle so suspended from a stirrup that it may rotate about both horizontal and vertical axes. In other words, this instrument is a dipping needle with automatic meridian adjustment. If the center of gravity coincides with the axis of rotation, the instrument will measure the inclination. If a counterweight, in the form of a small piece of wax, is attached to the needle and properly adjusted, the instrument will furnish vertical intensity anomalies.

The dip needle is a well known magnetic instrument for rapid observations. It consists of a magnetic pointer mounted in bearings so that it may rotate freely about a horizontal axis in a vertical plane.

Another instrument for the measurement of vertical intensities which has been used extensively in mining is the Thomson-Thalen magnetometer. It consists of a sensitive magnetic system rotating about a horizontal axis in a plane normal to the magnetic meridian. Mounted close to this system and underneath is a permanent magnet which is used to compensate for the normal vertical intensity effects. The position of this magnet may be changed by a micrometer-screw. Thus, a vertical intensity anomaly will appear as a reading of the distance of the compensating magnet from the magnetic system.

Magnetic instruments based on other principles have also been suggested, such as the magnetron, a two element vacuum tube whose plate current is critically dependent on an axial magnetic field. Another is the magnetic torsion-balance which is a modification of the Eoetvoes torsion-balance. A third example is the earth inductor gradiometer in which electromotive forces induced in two coils rotating about a horizontal axis are compared by a bridge arrangement, thus giving the horizontal-vertical intensity gradient. Modifications of the earth inductor have been suggested and used as sensitive prospecting instruments. Depending on the orientation of its axis of revolution, an earth inductor may be employed to measure any desired magnetic intensity component as long as the speed of revolution is kept constant or the intensity compensated. The constant speed principle utilizes a tuning fork and synchronous motor to drive the inductor. In the compensation instruments the intensity components to be measured may be reduced to zero by permanent magnets or by Helmholtz coils.

4. Error compensation.

In each of the precision magnetometers a number of local effects must be compensated or eliminated. These include the effects of temperature, magnetic variations during the measuring interval, planetary effects, proximity effects of magnetic objects, and terrain effects. Iron objects, power lines, and similar interference cannot be corrected for but must be avoided as much as possible.

CHAPTER II

THEORY

In March 1954, a new type magnetometer [4] using the principles of nuclear induction was fabricated at Varian Associates Research Laboratory, culminating development began in 1948. This device provided a means of measuring the magnitude of the earth's magnetic field in terms of the Larmor frequency of precession of nuclei possessing a spin with coupled magnetic moment.

1. Description of nuclear magnetic resonance.

The mechanism relating this precessional frequency to the strength of the external magnetic field is analogous to the effect of a strong fixed magnetic field acting on a bar magnet which is caused to spin about an axis parallel to its long dimension. The spinning magnet will precess like a top about the field with an angular frequency determined by the magnitude of the external field; i.e.,

$$\omega = \gamma_p H$$

in which

$$\gamma_p = \text{gyro-magnetic ratio} = \frac{\text{magnetic moment}}{\text{angular momentum}}$$

For protons in water γ_p has been measured to an absolute accuracy of better than one part in 100,000. In the free presession magnetometer a strong polarizing field is employed to orient the protons contained in the sample in a direction approximately perpendicular to the earth's magnetic field. When this polarizing field is suddenly removed, the earth's magnetic field causes the nuclear magnetic moment of the protons to precess about the lines of magnetic force at the Larmor frequency.

This precessing moment causes an induced voltage at this frequency to generate a current in a pickup coil. This signal is amplified and, by means of scaling circuits, analoged into a voltage proportional to the magnitude of the earth's magnetic field. The inherent accuracy of the system is quite high because the precessional frequency depends on the magnitude of the external field and the gyromagnetic ratio constant of the sample and nothing else. With present equipment one can obtain an accuracy of measurement of better than one part in 200,000. In Palo Alto, California, the nominal value of the earth's magnetic field is 50,000 gamma; hence, this corresponds to an accuracy of one-fourth gamma. Temperature compensation is not a problem because this precessional frequency will change by less than one part in three million for a 100 degree temperature change.

It should be emphasized that the precessional frequency of the received signal depends only on the total magnitude of the external magnetic field. If the polarizing field is not oriented perpendicular to the magnetic field, the initial amplitude of the received signal will be reduced but the frequency will remain the same. Consequently, the high accuracy of the measurement will be maintained regardless of the orientation of the polarizing field.

It was felt that by use of a biasing field fixed in a reference plane and of suitable magnitude, the basic magnetometer could be adapted to the measurement of the direction and components of the earth's magnetic field. It was found that the biasing field would be of smaller magnitude than the earth's field in order to minimize the effect of assumptions made in the formula derivations. Further, since the measurement of the absolute magnitude of such a small field would be very

difficult, development of the variometer was undertaken assuming only approximate knowledge of the size of the biasing field. The manner in which the biasing field is used will be apparent from the following derivations.

2. Derivation of formula for measurement of declination.

Consider three coplanar vectors F, F^+, and F^-, which represent the total earth's field, the resultant field with the application of a positive biasing field, and the resultant field with the application of a negative biasing field, respectively. The biasing fields are represented by vectors A^+ and A^-, equal in magnitude and opposite in direction. As shown in Figure 1, the direction of the vector F is assumed to differ from the axis of the biasing field coils by an angle $\Delta\phi$. Then, from the law of cosines:

$$F^{+2} = F^2 + |A|^2 - 2|A| F \cos\phi \tag{1}$$

$$F^{-2} = F^2 + |A|^2 - 2|A| F \cos\theta \tag{2}$$

$$\cos\phi = -\cos\theta \tag{3}$$

$$\cos\phi = \cos\left(\Delta\phi + \frac{\pi}{2}\right) = -\sin\Delta\phi$$

Subtracting (2) from (1)

$$F^{+2} - F^{-2} = -4|A| F \cos\phi \tag{4}$$

Or

$$|A| = \frac{F^{+2} - F^{-2}}{-4F\cos\phi} = \frac{F^{+2} - F^{-2}}{4F\sin\Delta\phi} \tag{5}$$

Adding (1) and (2)

$$F^{+2} + F^{-2} = 2F^2 + 2|A|^2 \tag{6}$$

Substituting (5) into (6)

$$F^{+2} + F^{-2} = 2F^2 + \frac{2(F^{+2} - F^{-2})^2}{16\, F^2 \sin^2 \Delta\phi}$$

$$\sin^2 \Delta\phi = \frac{(F^{+2} - F^{-2})^2}{8F^2[F^{+2} + F^{-2} - 2F^2]}$$

$$\sin \Delta\phi = \frac{(F^{+2} - F^{-2})}{2\sqrt{2}\, F\,[F^{+2} + F^{-2} - 2F^2]^{1/2}} \tag{7}$$

Now let

$$F^{+} = F + \Delta F^{+}$$
$$F^{-} = F + \Delta F^{-}$$

Substituting these into (7)

$$\sin\Delta\phi = \frac{(\Delta F^{+} - \Delta F^{-})(\Delta F^{+} + \Delta F^{-} + 2F)}{2\sqrt{2}\, F\,[(2F + \Delta F^{-})(\Delta F^{-}) + (2F + \Delta F^{+})(\Delta F^{+})]^{1/2}}$$

$$\sin\Delta\phi = \frac{(\Delta F^{+} - \Delta F^{-})(\frac{\Delta F^{+}}{F} + \frac{\Delta F^{-}}{F} + 2)}{2\sqrt{2F}\,[(2 + \frac{\Delta F^{-}}{F})(\Delta F^{-}) + (2 + \frac{\Delta F^{+}}{F})(\Delta F^{+})]^{1/2}} \tag{8}$$

Now assume that $\frac{\Delta F^{+}}{F} << 1$ and $\frac{\Delta F^{-}}{F} << 1$

Then (8) can be simplified to

$$\sin\Delta\phi = \frac{\Delta F^{+} - \Delta F^{-}}{2[F(\Delta F^{+} + \Delta F^{-})]^{1/2}} \tag{9}$$

Finally, it is assumed that $\Delta\phi$ is a small angle so that $\sin\Delta\phi = \Delta\phi$.

This reduces (9) to

$$\Delta\phi = \frac{1}{2}\,\frac{\Delta F^{+} - \Delta F^{-}}{[F(\Delta F^{+} + \Delta F^{-})]^{1/2}} \tag{10}$$

In order to refer variations of the coplanar angle to the variations of declination in the horizontal plane, consider Figure 2.

$$b^2 = f^2 + d^2 \tag{11}$$

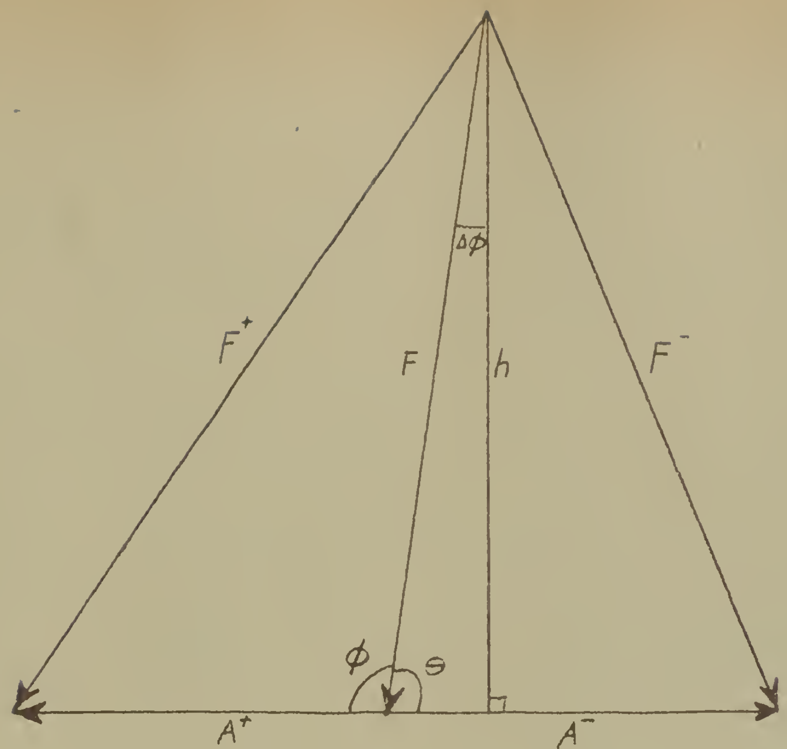


Figure 1
Vector Diagram for Declination Measurement

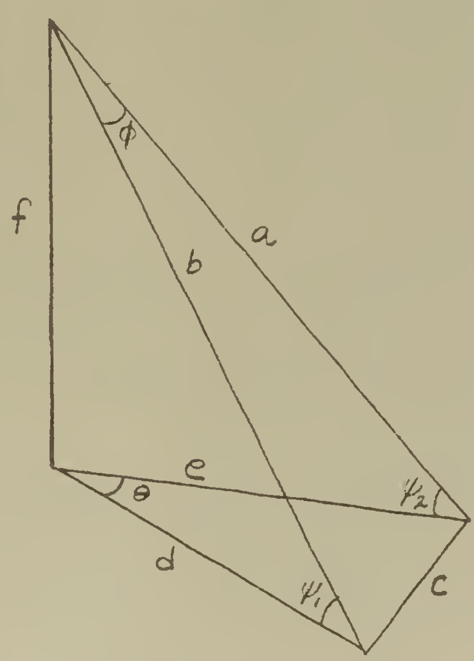


Figure 2
Relation Between Ø and θ

$$a^2 = f^2 + e^2 \tag{12}$$

$$c^2 = a^2 + b^2 - 2ab\cos\phi \tag{13}$$

$$c^2 = d^2 + e^2 - 2de\cos\theta \tag{14}$$

Substituting (11) and (12) into (13) and (14)

$$c^2 = 2f^2 + d^2 + e^2 - 2ab\cos\phi \tag{15}$$

$$c^2 = d^2 + e^2 - 2de\cos\theta \tag{16}$$

Subtracting (16) from (15) and solving for $\cos\phi$

$$2f^2 - 2ab\cos\phi + 2de\cos\theta = 0$$

$$ab\cos\phi = f^2 + de\cos\theta$$

$$\cos\phi = \frac{f}{a}\cdot\frac{f}{b} + \frac{d}{b}\cdot\frac{e}{a}\cos\theta$$

$$\cos\phi = \sin\psi_1\sin\psi_2 + \cos\psi_1\cos\psi_2\cos\theta \tag{17}$$

This equation may now be applied to (10) by letting $a = h$, $b = F$, $\phi = \Delta\phi$

and assuming: $\psi_1 = \psi_2 = I$, the inclination

Substituting these approximations into (17)

$$\cos\Delta\phi = \sin^2 I + \cos^2 I\cos\Delta\theta$$

$$\cos\Delta\phi = 1 - \cos^2 I + \cos^2 I\cos\Delta\theta$$

$$\cos\Delta\phi = 1 + (\cos\Delta\theta - 1)\cos^2 I$$

$$\cos\Delta\phi = 1 - 2\cos^2 I\sin^2\left(\frac{\Delta\theta}{2}\right) \tag{18}$$

Now since $\Delta\phi$ and $\Delta\theta$ are both so small, let

$$\cos\Delta\phi = 1 - \frac{(\Delta\phi)^2}{2}$$

$$\sin\left(\frac{\Delta\theta}{2}\right) = \left(\frac{\Delta\theta}{2}\right)$$

Hence, (18) becomes

$$1 - \frac{(\Delta\phi)^2}{2} = 1 - 2\cos^2 I\left(\frac{\Delta\theta}{2}\right)^2$$

Or

$$\Delta\phi = \cos I\,\Delta\theta \tag{19}$$

Substituting (19) into (10) we have

$$\Delta\theta = \frac{(\Delta F^+ - \Delta F^-)}{2\cos I\sqrt{F}\sqrt{\Delta F^+ + \Delta F^-}} \tag{20}$$

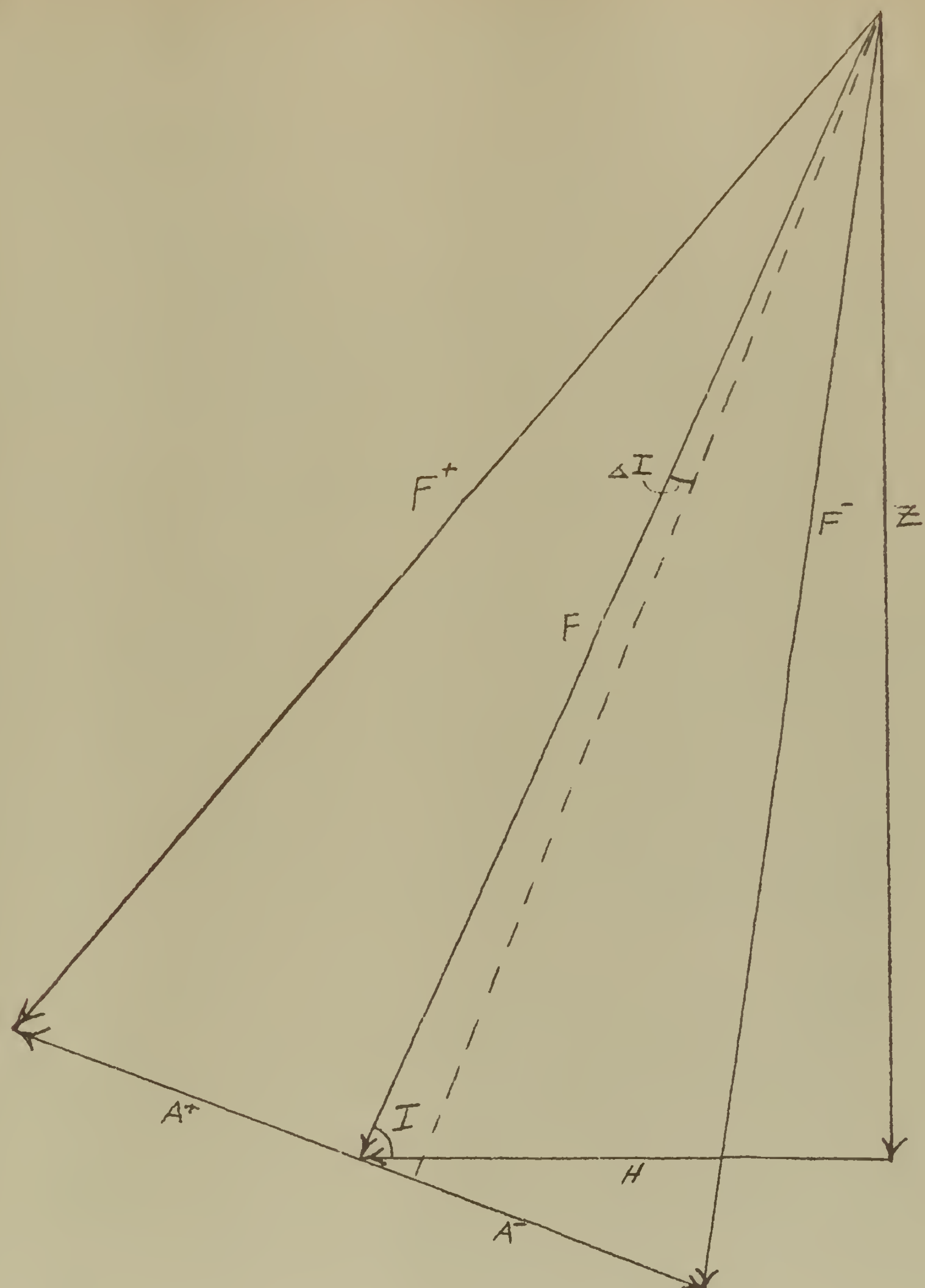


Figure 3
Vector Diagram for Inclination Measurement

At Palo Alto, California, the following values may be assumed essentially constant for the conditions under which the measurements were taken:

$$F = 51,350\ \gamma$$

$$I = 62^\circ$$

$$\Delta F^+ + \Delta F^- = 700\ \gamma$$

Substituting these into (20) gives the relation between variations in declination and the magnitudes of ΔF^+ and ΔF^- as measured by the magnetometer

$$\Delta D = 1.78 \cdot 10^{-4} (\Delta F^+ - \Delta F^-)$$

3. Derivation of formula for measurement of inclination.

The formula for the measurement of the variations of the angle of inclination may be derived in a manner similar to that of declination. In this case it is merely necessary to orient the axis of the biasing field coils in the plane of the magnetic meridian and approximately perpendicular to the earth's magnetic field. Then, from Figure 3, it can be seen that

$$\Delta I = \left[\frac{1}{2\sqrt{F}\ \sqrt{\Delta F^+ + \Delta F^-}} \right] (\Delta F^+ - \Delta F^-)$$

We now have means for measuring the magnitude and direction of the variations of the earth's magnetic field and so can completely determine it. From this information the variations of the horizontal and vertical components can be found if required.

4. Design of bias field coils.

The problem of providing a suitable biasing field required by the foregoing derivations was solved by making several compromises. The

Larmor frequency formula for the precessional frequency assumes that the field across the sample area is uniform. If the field were not uniform, the various magnetic moments would precess at slightly different rates. This phase incoherence would ultimately destroy the received signal amplitude. The first compromise, then, was the determination of the required field uniformity across the sample volume.

Bloch [2] has shown that for a Lorentzian line shape the half line width ΔH in gauss is given by

$$\Delta H = \frac{1}{\gamma_p T_2}$$

where T_2 is the characteristic transverse relaxation time. Under the conditions of the experiment this time was about one second. Substituting this value into the above formula and solving for ΔH, we have a half width value of 3.7 gamma. This is the allowed variation of the external magnetic field across the sample volume.

Now, the general expression for the total field in the diagram below is:

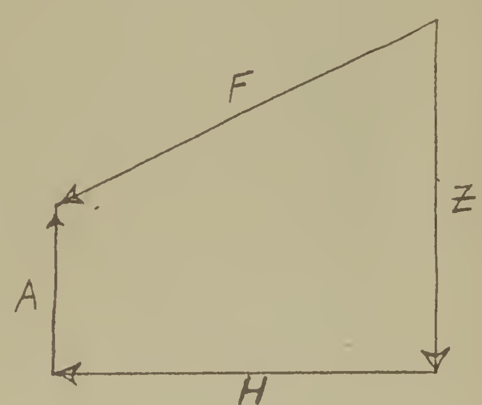
$$F^2 = A^2 + Z^2 + H^2 - 2AZ$$

Hence,

$$dF = \frac{\partial f}{\partial H}\, dH + \frac{\partial f}{\partial Z}\, dZ$$

Substituting gives,

$$A = \frac{F\,dF - H\,dH - Z\,dZ}{-dZ}$$



Now, assume that dH and dZ are each one gamma changes and a change dF of one-third gamma is to be detected. The value of the required bias field A as determined by substituting these conditions into the above formula is 6700 gammas. Hence, the ratio H/A = 3.7/6700 is the required field uniformity of the bias field, approximately one part in two thousand.

Knowing the required field uniformity, it is now necessary to determine the configuration of the biasing field coils. This is done in Appendix I by expressing the field of one, two, and three coil combinations in the form of an infinite series of spherical harmonics. The field uniformity for each is then simply the ratio of the first error term and the constant field term along the radial direction. Garrett [7] and Blewett [1] provide a thorough treatment of similar configurations.

The results of these three cases are as follows:

a) Single coil case:

The ratio of the constant field B_r to the error term ΔB_r is

$$\frac{B_r}{\Delta B_r} = \frac{(-\frac{2}{3})\, P_1(\cos\theta)}{(\frac{r}{a})^2\, P_3(\cos\theta)} = \frac{2000}{1}$$

Solving for the coil radius for $\theta = 0$,

$$a = 54.7\ r$$

where r is the radius of the sphere containing the sample.

b) Two coil case:

$$\frac{B_r}{\Delta B_r} = \frac{P_1'(\cos\alpha)\ P_1(\cos\theta)}{(\frac{r}{a})^4\ P_5'(\cos\alpha)\ P_5(\cos\theta)} = \frac{2000}{1}$$

Solving for the coil radius as before

$$a = 7.74\ r$$

c) Three coil case:

$$\frac{B_r}{\Delta B_r} = \frac{2.1\ P_1(\cos\theta)\ a^7}{(-4.97)\ P_7(\cos\theta)\ a\ r^6} = \frac{2000}{1}$$

Again, solving for the coil radius gives

$$a = 4.1\ r$$

The effective sample size used was about two inches in radius. This would require biasing field coils of radii 109 inches, 15.5 inches, and

8.2 inches, respectively for the above three cases. From a consideration of the material required and ease of construction, the two coil Helmholtz pair was selected.

Reference to McComb [10] gives the design formula for Helmholtz coils as

$$A = \frac{89.9\ NI}{a}$$

I in Ma
A in gamma
a in cm

from which it was found that 190 milliamperes of current would be required after arbitrarily selecting 17 turns of wire per coil.

CHAPTER III

EXPERIMENTAL EQUIPMENT

1. Description of the free precession magnetometer.

The free precession magnetometer consists of two physically separated units. The first of these is the magnetic field sensing unit whose function is to produce polarization of an enclosed sample and to pick up the precession frequency corresponding to the strength of the magnetic field being measured. The second unit is the frequency measuring equipment.

The magnetic field sensing unit is a quart size glass jar, filled with ordinary tap water, and surrounded by a solenoidal coil. This coil is used for both polarizing the enclosed sample and picking up the precession signal by means of a relay switching unit. A Faraday shield is mounted on the coil to minimize the problem of floating grounds.

The frequency measuring unit consists essentially of scaling circuits. The frequency of the exponentially decaying precession signal is measured by counting 2048 cycles of the signal in the slow counter. The time required for this measurement is then applied as a gate pulse to the fast counter. This counter responds to the number of cycles of a 100 kcs frequency standard signal occuring within this interval. This information from the fast counter is analoged into a voltage by use of a bank of relay operated mercury batteries. This voltage is measured by a pen recorder.

Using a counting interval of about one second and a frequency standard of 100 kcs, the estimated accuracy of the magnetometer is one-half gamma.

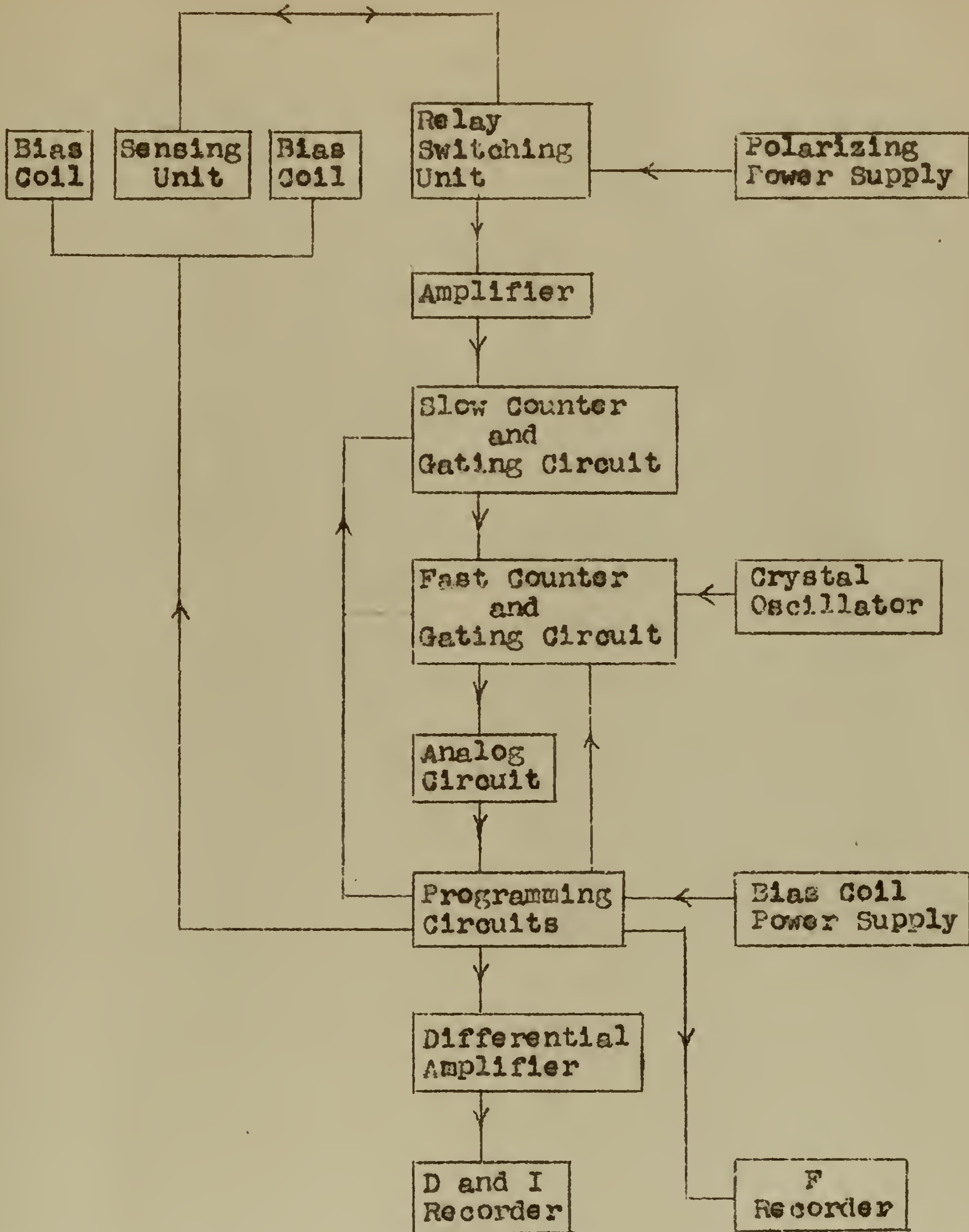


Figure 4
Block Diagram of Declinator

2. Description of the declinator.

The magnetometer described above was modified for making measurements of declination as shown in Figure 4. The biasing coils were placed at the Helmholtz conditions in order to provide a uniform magnetic field at right angles to the earth's field. These Helmholtz coils were constructed with a radius of 17 inches and were wound on plywood sheets which had been glued together cross grained in order to minimize warpage. Each coil consisted of 17 turns of number 18 enameled copper wire in three layers. Three aluminum separating rods were made to maintain these coil forms at a distance of 17 inches with aluminum lock washers. These one inch rods were spaced every 120 degrees about a circle of radius 15 inches from the coil center.

The field uniformity given by this arrangement was tested by measuring the T_2 of the received signal with and without the bias field applied. The results of several tests indicated that the biasing field was sufficiently uniform so as to have little effect on the T_2 out to a radius of five inches from the center of the coil axis. In the final arrangement the sensing unit was supported in the center of the bias field coils and coaxial with them.

The programming circuit consisted of eleven cam operated microswitches driven by a two rpm synchronous motor. These microswitches were connected as shown in Figure 5 and provided the synchronization between the various units of the declinator. This unit, then, allowed the same magnetometer circuits to be used for the measurement of F, F^+, and F^-. Figure 6 is a time sequence diagram which indicates how these various functions were controlled. The polarizing field was energized for a

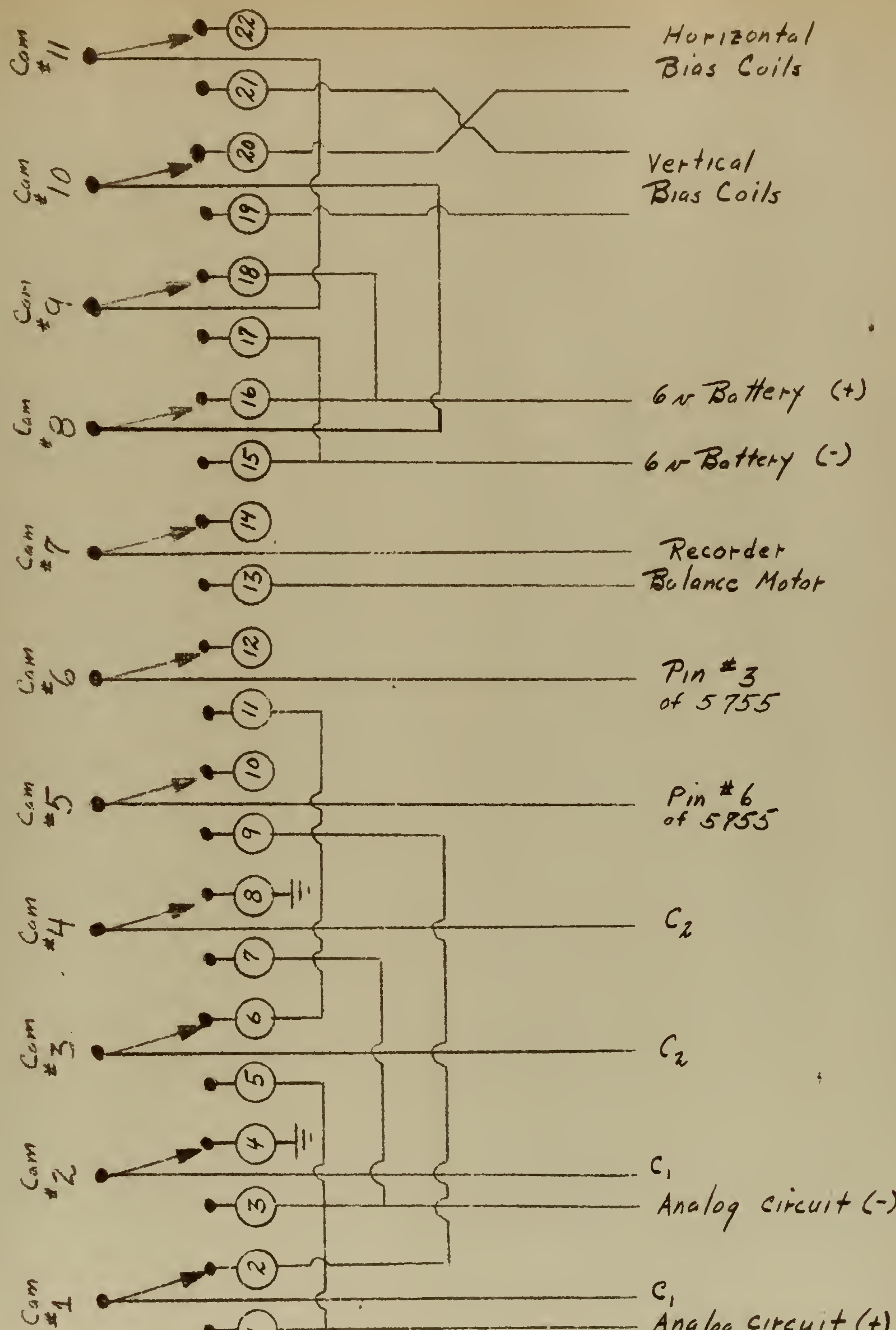


Figure 5
Programming Circuit Cam Connections

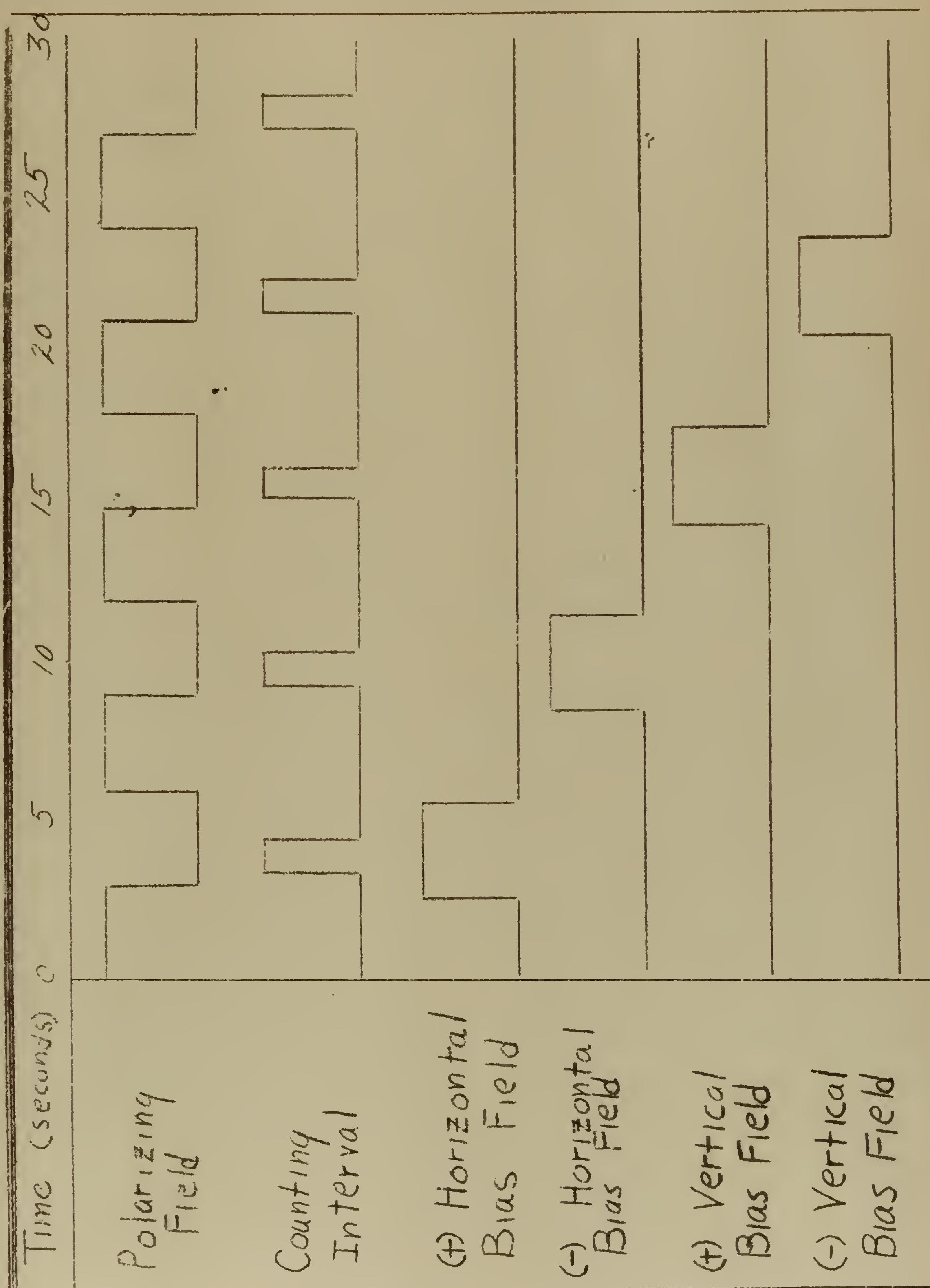


Figure 6
Programming Circuit Time Diagram

period of three seconds every six seconds. The precession signal was received during the period when the polarizing field was deenergized. The interval for counting the frequency of this signal started shortly after the collapse of the polarizing field to allow for the decay of switching transients. The horizontal bias field, arbitrarily marked plus, lasted over the first counting interval so that the magnitude of the resultant magnetic field F^+ was measured. During the second counting interval, the field F^- was measured. This cycle was again repeated during the third and fourth receiving intervals although provision was made to use this period for making F^+ and F^- measurements in the plane of the magnetic meridian for measuring inclination. The fifth counting interval was used to measure the magnitude of the earth's field with no bias field applied. The sequence of operations then provided two measurements of declination and one measurement of the magnitude of the earth's field every 30 seconds.

Voltage information from the analog circuit was also programmed through this circuit to a differential amplifier. Since it was shown in Chapter II that variations in declination are proportional to the difference between F^+ and F^-, the output of this amplifier is a direct measure of declination. A detailed description of the operation of this circuit follows in the next section.

A curve of the variation of declination about an arbitrary reference line determined by the orientation of the bias field coils was plotted on the D and I recorder. This was a Speedomax pen recorder having a full scale deflection of ten millivolts with a paper feed set for five inches per hour.

3. Differential amplifier.

First of all it was necessary to devise a memory device for storing the analog voltage for one bias field while the other bias field information was being obtained. As shown in Figure 7, computer type plastic capacitors having very high leakage resistance were used as the storage device. When charged to the analog voltage, they were simultaneously applied to identical resistor networks which provided a high impedance discharge circuit with a time constant of about 110 seconds. The voltages developed across the 150 K grid resistors were applied to the two control grids of a type 5755 twin triode. This tube was selected because it has an excellent long time drift stability.

The voltage from the analog circuit varied from zero to 42.75 volts, and so the difference voltage across the grid resistors had a range of 85.5 volts. For this to be attenuated to a variation of .01 volts required a reduction of 8550. After some attenuation in the grid circuit and the tube amplification less than unity, the proper signal swing was achieved by a five to one attenuator between the two cathodes. This arrangement also met the requirement of a shunting impedance less than 1000 ohms across the chopper input circuit of the recorder. During the actual measurement runs the grid to ground resistance was increased to 180 K in order to give a slight increase in scale sensitivity. This arrangement required a voltage change from the analog circuit of 1.76 times its full scale value to give exactly full scale deflection on the declination recorder.

Low drift was a prime consideration in the operation of this circuit. A 6.3 volt storage battery was used as the filament supply source. Shielding the vacuum tube further increased the circuit stability by

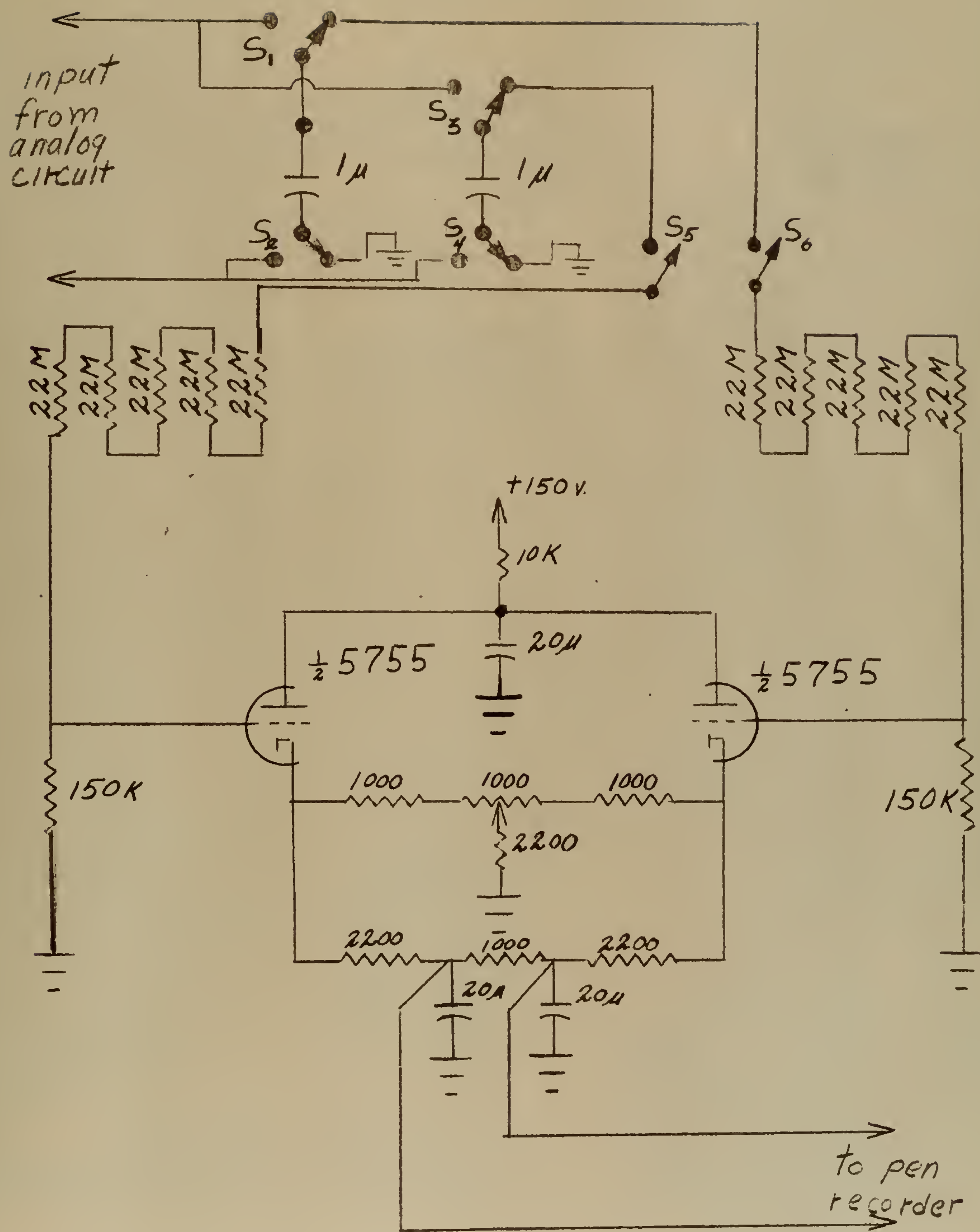


Figure 7
Differential Amplifier Time Diagram

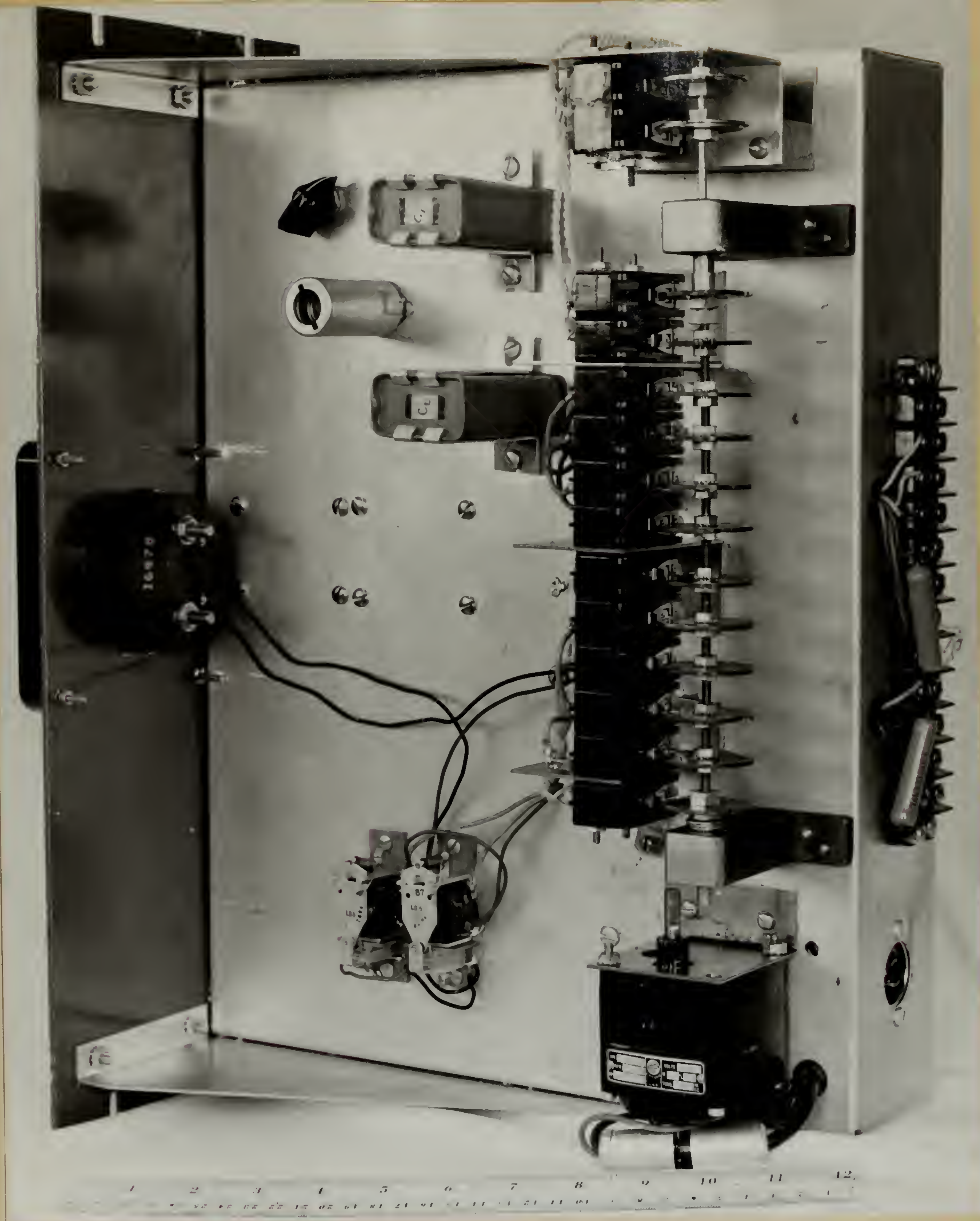


Figure 8
Programming Chassis

reducing the effect of stray fields. The use of negative feedback in the cathode circuit, and a regulated power supply, further enhanced the stability. Circuit linearity was tested by varying the input voltages in five volt steps throughout the range and noting the resulting deflection on the pen recorder. The circuit output varied less than one millivolt in 15 hours with the grids grounded.

A 1000 ohm potentiometer was placed in the cathode circuit in order to balance the difference of tube characteristics between the two halves of the 5755. It also served as a positioning control for the trace on the pen recorder.

A time diagram showing the sequence of operation of the switches in this circuit is shown in Figure 9. Switches S_1 and S_2 apply analog voltage to C_1 corresponding to F^+. Switches S_3 and S_4 apply analog voltage to C_2 corresponding to F^-. S_5 and S_6 simultaneously apply these voltages to the grid circuits of the differential amplifier. The switch S_7 closes the circuit to the balance motor of the recorder after the switching transients from S_5 and S_6 have decayed. Switches S_8 and S_9 control the bias field current.

When the entire programming circuit was first placed in operation the results were unsatisfactory because of switching transients. The source of these transients was determined to be 60 cycle pickup on the control grid circuit. A slight improvement was obtained by shielding the entire control grid circuit and the use of shielded leads up to the base of the tube. Finally, the addition of two 20 microfarad electrolytic capacitors connected from the recorder input leads to ground reduced these transients to less than one-tenth millivolt.

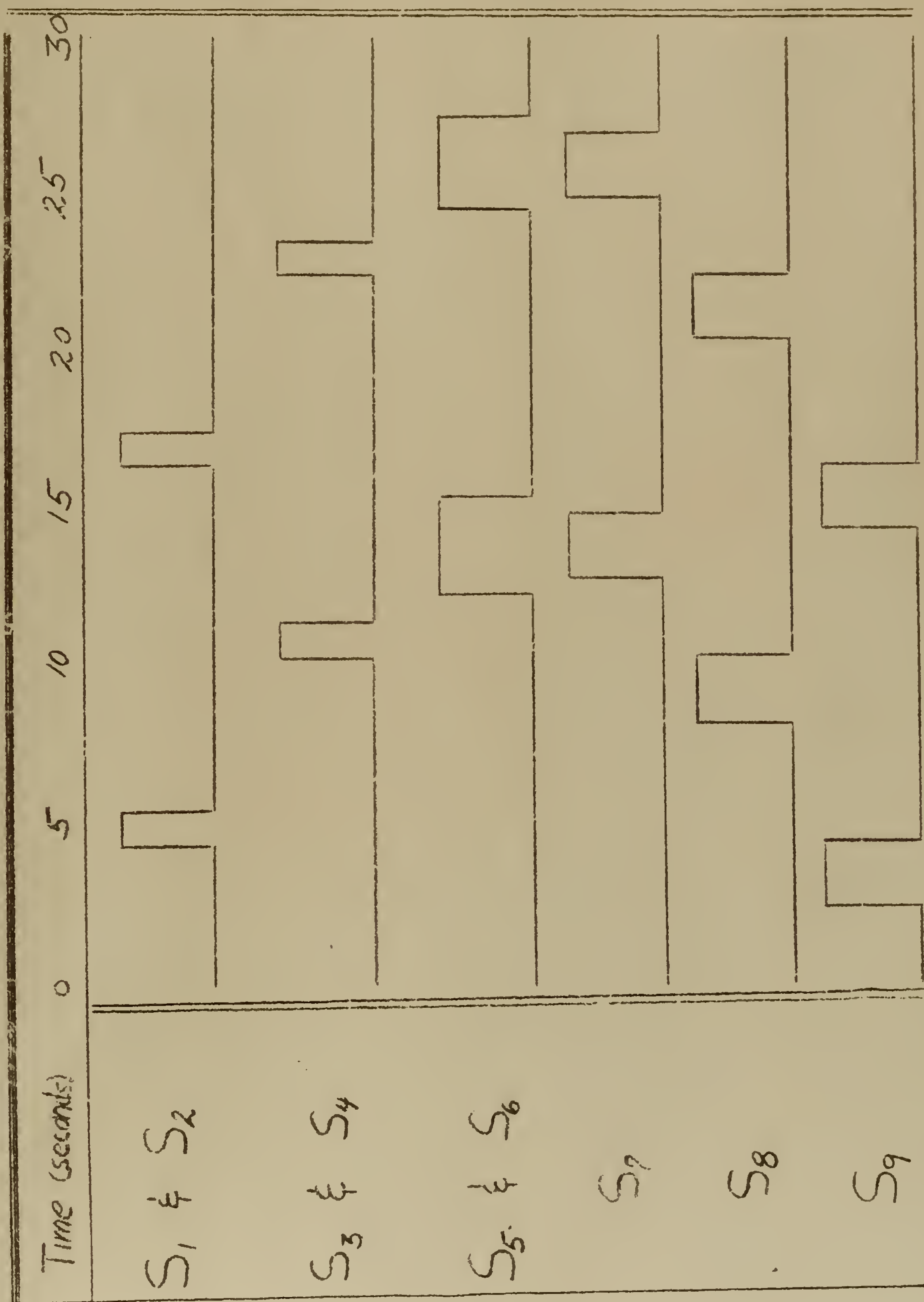


Figure 9
Differential Amplifier Time Diagram

The placement of grounds also proved to be of assistance in eliminating the effect of switching transients. It was found to be quite important to ground the recorder case to the amplifier chassis and also ground these units to the common magnetometer ground.

CHAPTER IV

INVESTIGATION

1. System adjustment.

After a satisfactory operational test had been made on the programing circuit, the equipment was integrated into the magnetometer circuit as shown in Figure 4. The bias field coils were assembled using one inch aluminum rod spacers with aluminum lock nuts so that the coils were parallel at a distance of 17 inches. The axis of the bias coils was oriented approximately perpendicular to the magnetic meridian and a few milliamperes of current passed through the coils first in one direction and then the other. The resulting F^{+} and F^{-} magnitudes were measured on the recorder, and then the coil axis readjusted until the measured magnitudes of these vectors were equal to within five gamma. The bias coil current was increased in this manner and the position of the coils readjusted until the computed value of 190 milliamperes was reached. The bias field coil forms were then securely wedged into place and the sensing unit for the magnetometer positioned as nearly as possible in the center of the two coils and coaxial with them.

To increase the stability of the system the differential amplifier was allowed to run continuously with B plus supplied from a regulated power supply and the six volt filament power supplied from a storage battery. Since the filament drain was approximately 350 milliamperes, a battery charger was floated across the battery to compensate for the power drain.

The bias field current was also supplied from a six volt storage battery since the system sensitivity depends in part upon the magnitude of the bias field. At first, the switching of the bias fields was done

Figure 10
Bias Field Coils

entirely by cam actuated microswitches, but it was found that the contact resistance of these microswitches varied too much to give accurate results. The total resistance in the bias coil circuit was about 29 ohms plus the internal resistance of the battery. Therefore, the variation of a fraction of an ohm in contact resistance of the microswitch to current flow in the positive and negative bias field conditions would cause a large error on the declination recorder. As a result, these microswitches were finally used to control two double pole single throw 30 volt dc relays which gave a much more constant contact resistance. Also, one side of this bias coil circuit was grounded to the programing circuit chassis to minimize the effect of 60 cycle pickup.

Slight adjustments of the fast and slow reset cams were required in order to obtain a good one second counting period which started just after the decay of the polarizing relay switching transient. Part of this difficulty was due to the fact that the five timing intervals on these cams were not spaced the same number of degrees apart since they were "hand made."

In order to check the amount of circuit drift, a double pole single throw dc relay was added to the circuit. This relay grounded the high side of the grid voltage divider resistors for a period of one minute every half hour. The reference markers, then, formed a base line which was later found useful in scaling down the declination curves.

Another difficulty encountered was the decrease in amplitude of the received signal when the bias field was applied. The cause of this was determined to be in the narrow band amplifier and preamplifier stages of the magnetometer. In order to obtain good signal to noise ratio the

bandwidth of the amplifier without the Q multiplier circuit installed was about 20 cps about a center frequency of 2190 cps. When the bias field was applied, the magnitude of the total field vector increased about 350 gammas. This caused the received signal to shift to 2204 cps. The amplifier and preamplifier tuned circuits were retuned to 2200 cps in order to place both the normal field and biased field signals on the same part of the response curve. This of course reduced the signal to noise ratio of the input circuits, but it was still sufficient under the conditions of counting interval and measurement period used. A cam operated stepping switch arrangement was designed for the purpose of retuning these amplifier circuits at center frequencies of 2190 cps and 2204 cps, but under the conditions of the experiment it was not found necessary to use it.

2. System calibration.

A small magnetic dipole was used to check the overall system operation. This dipole was fastened to a wooden turntable mounted on a motor geared to give one revolution every half hour. This dipole was located at a distance of 12 feet from the sensing unit along the bias coil axis and rotated in a plane inclined at an angle of 62 degrees. This orientation was necessary so that the inclination of the earth's magnetic field vector would not be affected by this artificial perturbation. Both the variations of the total field and the declination were recorded simultaneously.

It has been shown by McComb [10] that the far field of a magnetic dipole is not constant at a given distance for different orientations of

the dipole. However, the far field at a point on the magnetic axis is twice the field at a point the same distance along the perpendicular bisector of the magnetic axis.

The peak-to-peak value of the sine wave trace of the F recorder was 138 divisions, corresponding to 96.6 gammas. This means the value of the far field along the perpendicular bisector of the magnetic dipole was half this value, or 48.3 gammas. The perturbing field which caused the maximum declination variation was the field along the magnetic axis of the dipole. The magnitude of this field F_d was twice that along the perpendicular bisector, or 96.6 gammas.

The maximum variation of declination can now be computed from the formula:

$$\Delta D = \frac{F_d}{F \cos I}$$

Substituting appropriate values into this formula gives

$$\Delta D = \frac{96.6}{(51350) \cos 62^\circ}$$

$$\Delta D = 13.8 \text{ minutes}$$

The total change in declination is twice this or 27.6 minutes. Now, the peak excursion of the sinusoidal trace on the declination recorder was found to be 3 1/2 large scale divisions. Therefore, the system calibration is

$$\text{one large division} = \frac{27.6}{3.5} = 7.9 \text{ minutes of declination.}$$

This procedure was repeated at a distance of 18 feet and the results gave a system calibration of one large division equals 8.1 minutes of declination.

The equation for the variation of declination which was derived in Chapter II may also be used to check the scale calibration. Suppose the declination varies from zero to six minutes from the magnetic meridian. Then the difference between F^+ and F^- should be

$$(\Delta F^+ - \Delta F^-) = 2\,\Delta D \cos I \sqrt{F}\,\sqrt{\Delta F^+ + \Delta F^-}$$

Where:

$$F = 51350\ \gamma$$
$$\Delta F^+ + \Delta F^- = 700\ \gamma$$
$$I = 62^\circ$$
$$\Delta D = 6' = 1.75 \cdot 10^{-3}\ \text{radians}$$

Substituting these values

$$(\Delta F^+ - \Delta F^-) = 2(1.75)10^{-3} \cos 62^\circ \sqrt{(5.135)(7)10^6}$$
$$= 9.8\ \gamma$$

Now, since the F recorder full scale excursion is set for 70 gammas, this would correspond to 0.14 of full scale. The D recorder requires 1.76 of this variation for full scale deflection. Hence, a six minute change in declination will correspond to 0.08 of full scale on the declination recorder, or 0.8 large divisions. The system calibration is therefore:

$$\text{one large division} = \frac{6}{0.8} = 7.5 \text{ minutes of declination.}$$

The difference between the experimental and derived scale calibration values is well within the range of experimental error.

3. Discussion of system accuracy.

From a consideration of the formula derived for the measurement of declination it may be seen that the dominant source of error is due to the accuracy of measurement of F^+ and F^-. If it is assumed that the scaling

circuits can measure the precessional frequency to an uncertainty of one count, the magnitudes of F^+ and F^- can be measured to about one-half gamma. This follows from the fact that a 100 kcs reference frequency was counted for an interval of about one second. An uncertainty of one count thus corresponds to one part in 100,000.

In section 2 it was shown that a declination variation of six minutes would result in a difference of ten gammas between F^+ and F^-. If it is now assumed that either magnitude can be measured to an accuracy of one-half gamma, then the difference between these vectors can be measured to an accuracy of one gamma. The measurement will thus be accurate to ten per cent, or, in this case, to one-half minute of declination. It was determined that each large division on the recorder paper was about eight minutes of declination. Hence, this "count noise" of one-half minute of declination should be random fluxuations of 0.7 small scale divisions superimposed on the declination trace. The trace shown in Figure 9 illustrates the effect of this error.

A second source of error results from an assumption made in the derivation of the formula for the variation of declination. In Chapter II equation (9) was derived from equation (8) by assuming the $\frac{\Delta F^+}{F} \ll 1$ and $\frac{\Delta F^-}{F} \ll 1$. For convenience, let $\Delta F^+ = \Delta F^- = 350$ gammas, and $F = 51{,}000$ gammas. The ratio $\frac{\Delta F}{F}$ is therefore $\frac{350}{51000} = .0069$. Substituting these values in equations (8) and (9) and simplifying, it can be shown that this assumption introduces an error of 0.7 per cent.

Using the same equations as in the above paragraph it is found that this assumption also places a limit on the strength of the bias field which can be used. An error of ten per cent will result from use of a

biasing field with the same magnitude as the earth's field.

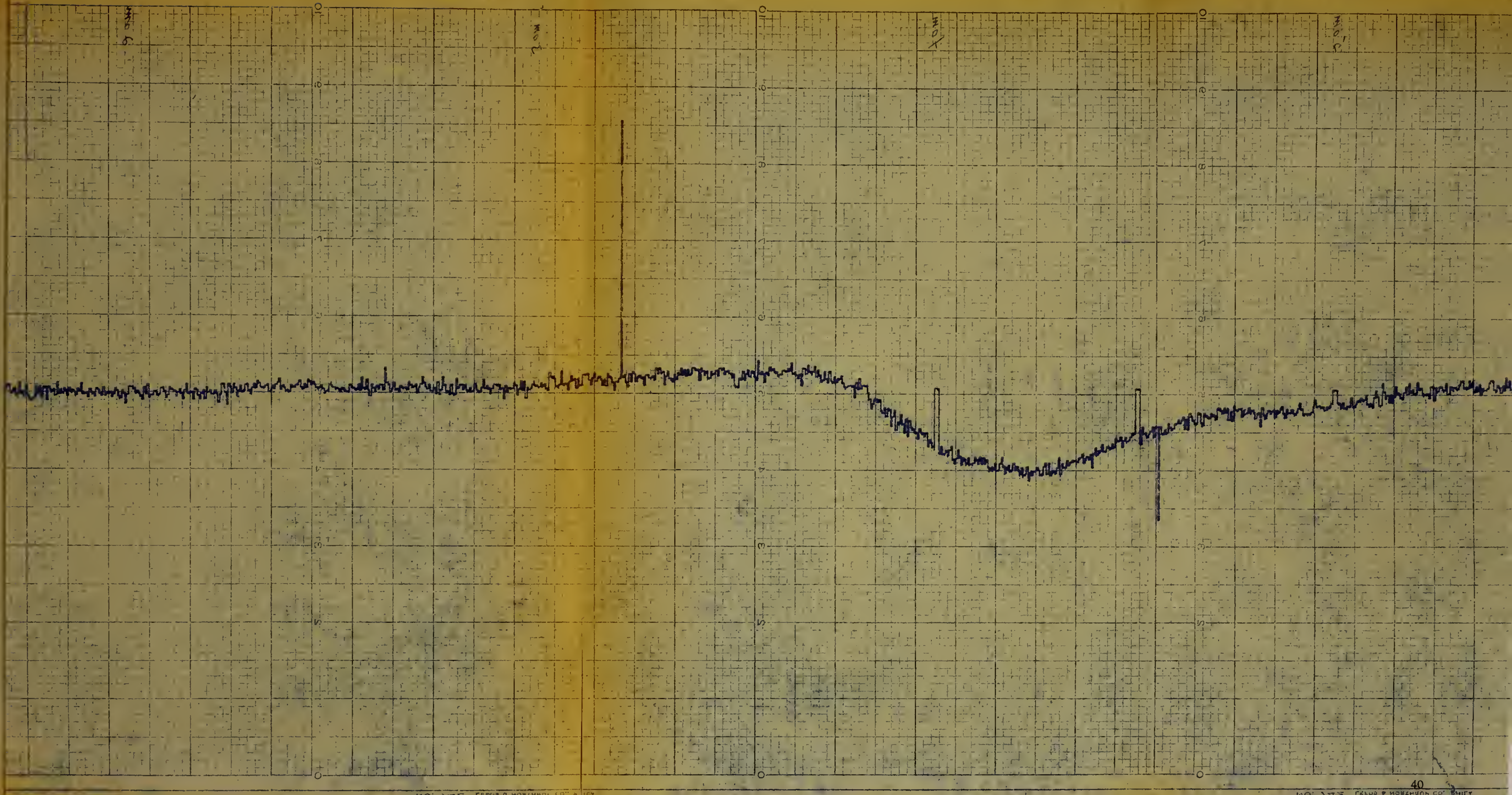
It should be noted, however, that the sensitivity of the system is proportional to the strength of the bias field. Increased sensitivity to small variations in declination will result from a stronger bias field. Hence, regulation of the current through these coils provides a convenient method for controlling the system sensitivity.

CHAPTER V

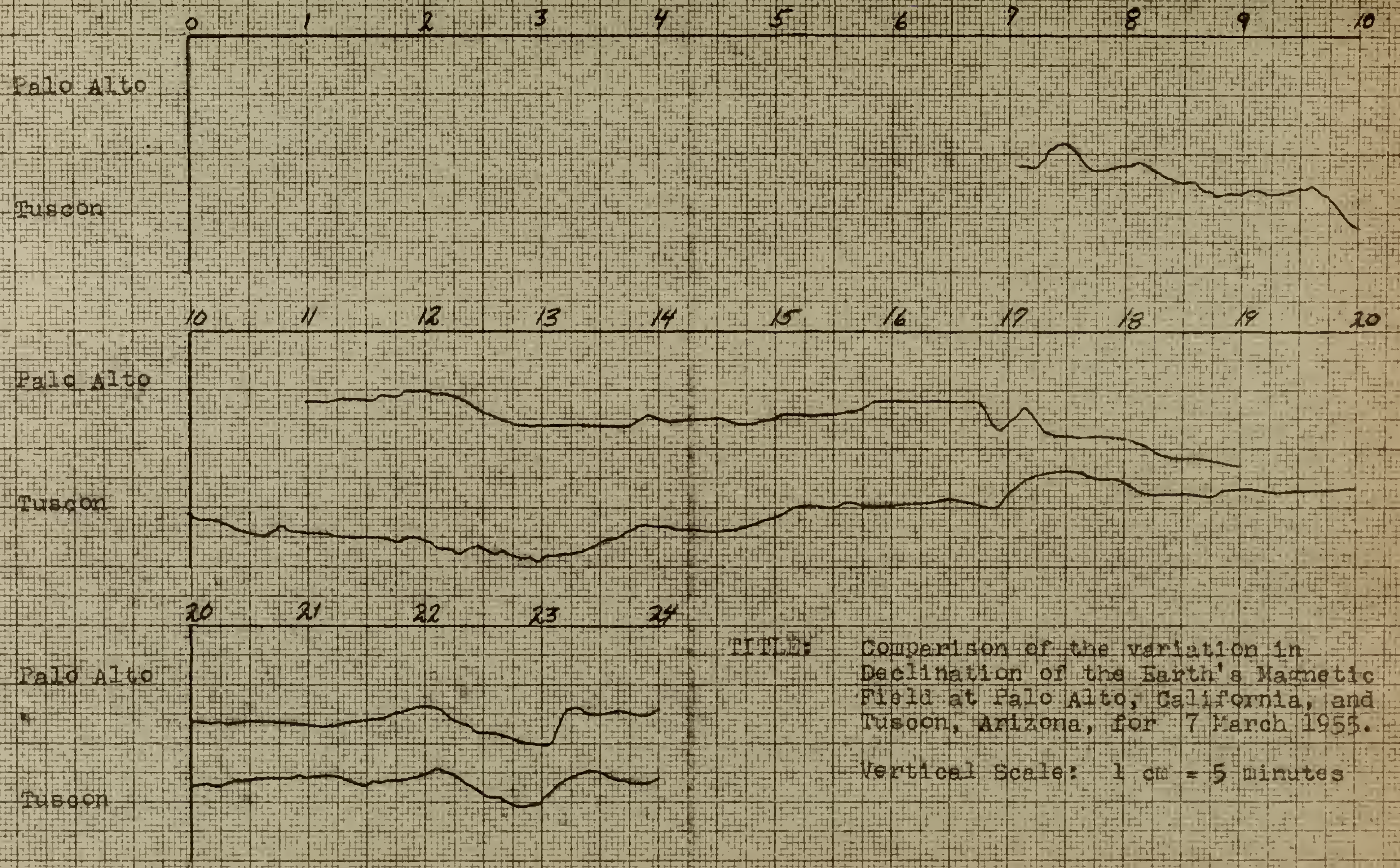
EXPERIMENTAL RESULTS

After several short runs the system was allowed to run more or less continuously from March 7 to March 13, 1955, and the variations of declination recorded on a Speedomax pen recorder at a time rate of five inches per hour. A reprint of an actual record is enclosed as Figure 11. The vertical scale of this trace is about 80 minutes of declination full scale. Some of the more rapid trace variations are due to the magnetometer "count noise" and switching transients. However, as an examination of the trace will show, periodic perturbations lasting several minutes are also present. These may have more significance when the sensitivity of the system is increased. Also, "time markers" occur every half hour which cause the trace to drop to the reference line of 5.1 on the recorder paper. This is the line along which there is no deviation of declination from the plane of the bias coils.

In order to correlate the results obtained with the photographic traces of declination received from the Magnetic Observatory, Tuscon, Arizona, the declination curves were replotted to a time scale of about two centimeters per hour. Both curves are plotted to the same scale, using Pacific Standard time, and the same vertical scale of one-half minute per millimeter. Except for local disturbances the diurnal variations of the earth's magnetic field should be about the same. This is seen to be the case from 1700 each evening until 0800 the next morning. During the daytime hours there were some departures between the two curves possibly due to local disturbances in the vicinity of the bias coils.



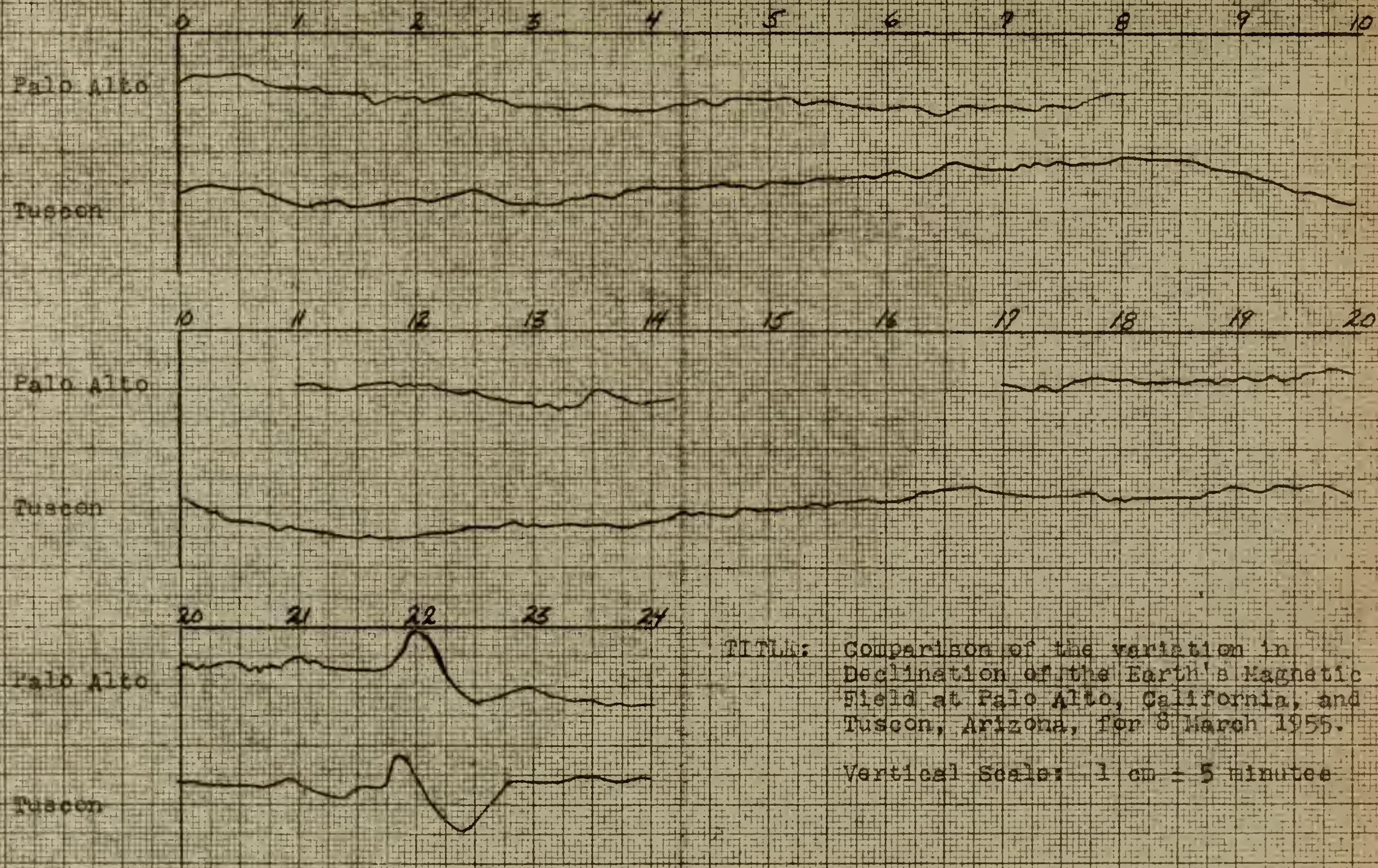
The value for inclination of 62 degrees used in this paper was also obtained experimentally with this system. As described in Chapter II, the bias field coils were oriented with their axis in the magnetic meridian. A few milliamperes of current were passed through the coils and the resultant shift of the F recorder noted. This current was then reversed and this second shift of the F recorder noted. The axis of the bias coils was then positioned until the difference between these two measurements was less than three gammas. The field current was then increased for better sensitivity in several increments to 190 milliamperes until negligible difference between the vector measurements remained. The angle between the plane of the bias field coils and the horizontal was then computed after measuring the sides of the triangle. It should be noted that for the above measurement the coil axis was perpendicular to the earth's magnetic field. For this coil orientation the variations of the angle of inclination could be recorded using the same magnetometer and difference amplifier arrangement as was employed to record declination. The formula which would be used to calibrate the equipment under such conditions is that noted in section 3 of Chapter II.



TITLE: Comparison of the variation in Declination of the Earth's Magnetic Field at Palo Alto, California, and Tuscon, Arizona, for 7 March 1955.

Vertical Scale: 1 cm = 5 minutes

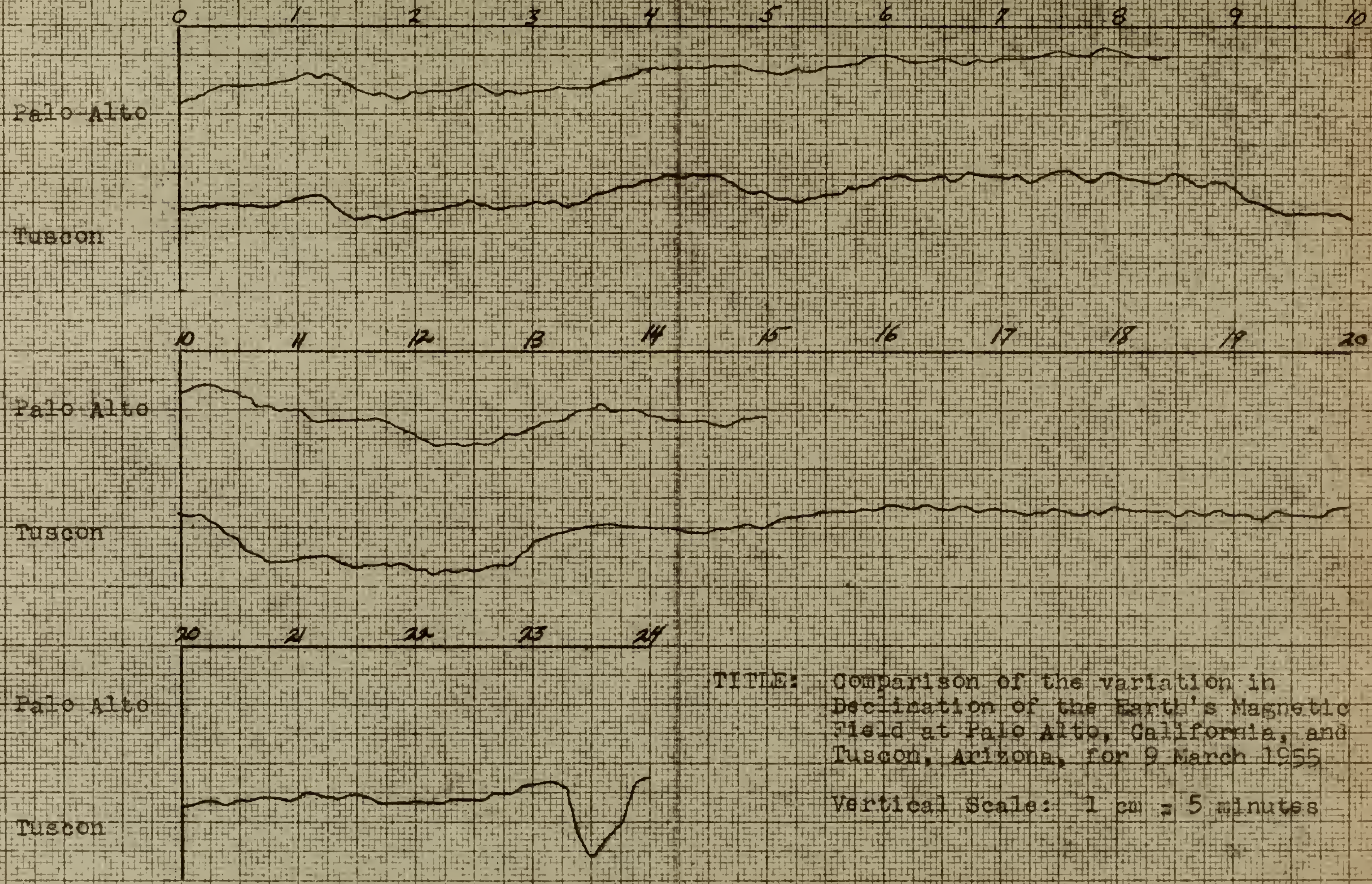
Figure 12



TITLE: Comparison of the variation in Declination of the Earth's Magnetic Field at Palo Alto, California, and Tuscon, Arizona, for 8 March 1955.

Vertical Scale: 1 cm = 5 minutes

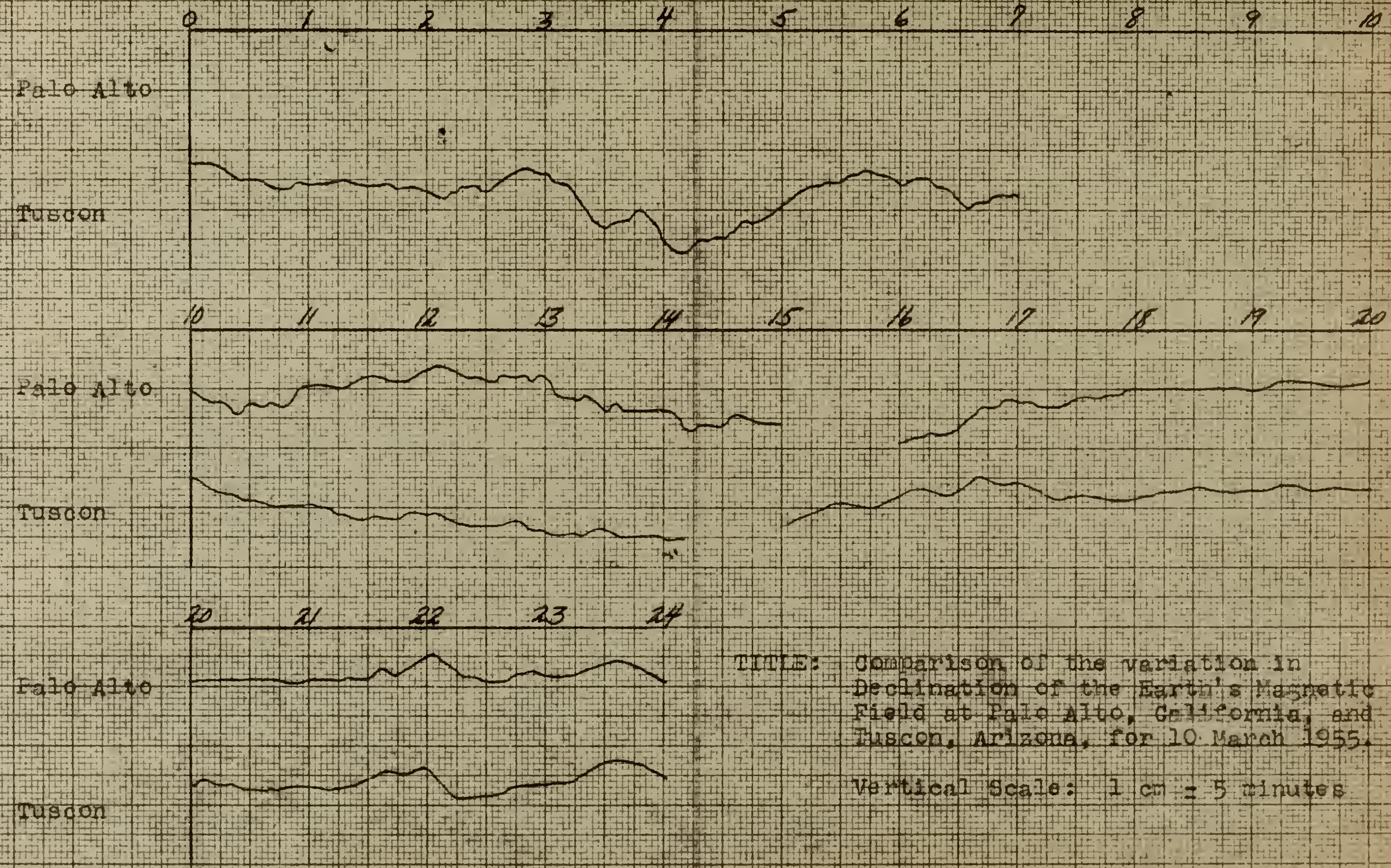
Figure 13



TITLE: Comparison of the variation in Declination of the Earth's Magnetic Field at Palo Alto, California, and Tuscon, Arizona, for 9 March 1955

Vertical Scale: 1 cm = 5 minutes

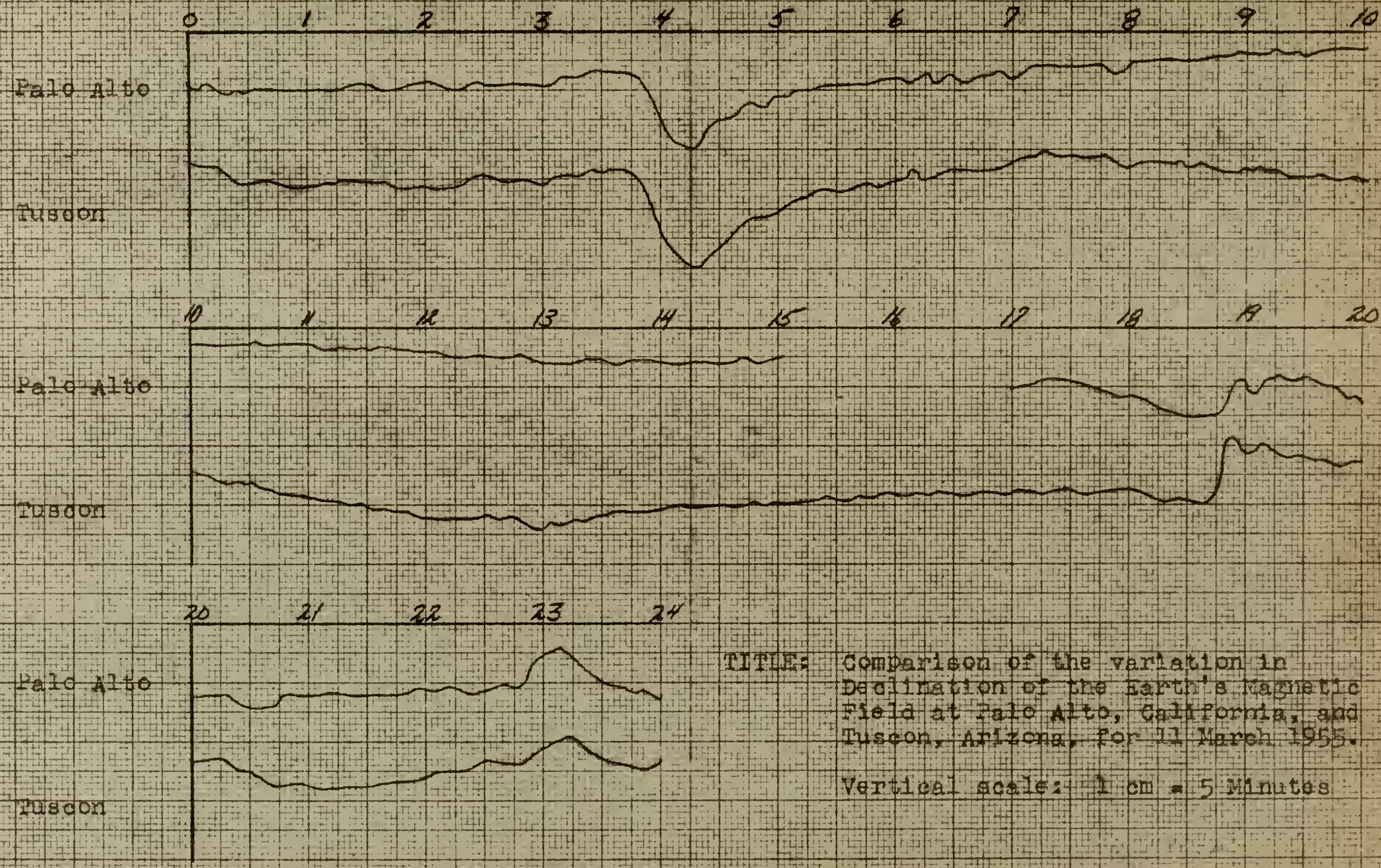
Figure 14



TITLE: Comparison of the variation in Declination of the Earth's Magnetic Field at Palo Alto, California, and Tuscon, Arizona, for 10 March 1955.

Vertical Scale: 1 cm = 5 minutes

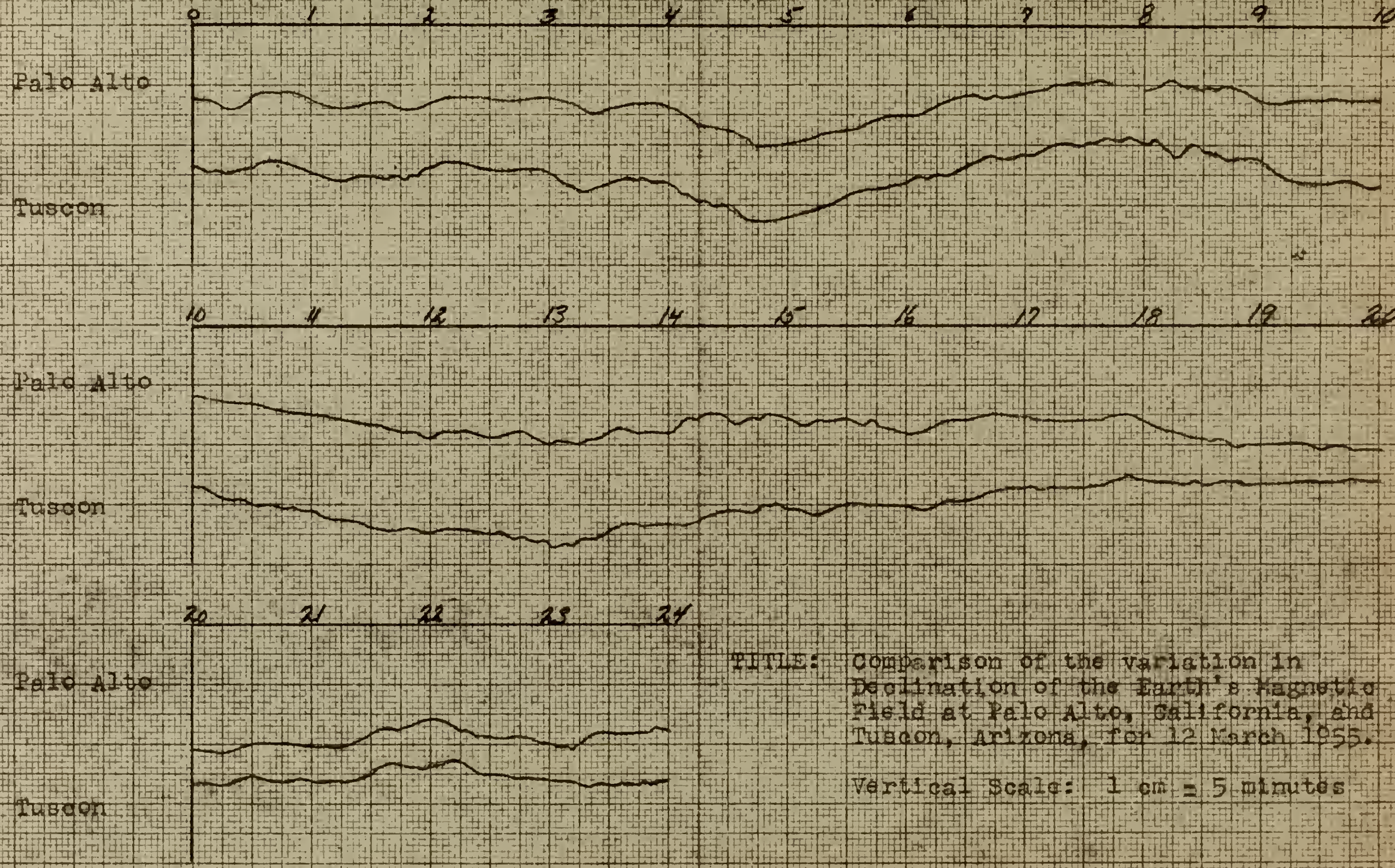
Figure 15



TITLE: Comparison of the variation in Declination of the Earth's Magnetic Field at Palo Alto, California, and Tuscon, Arizona, for 11 March 1955.

Vertical scale: 1 cm = 5 Minutes

Figure 16



TITLE: Comparison of the variation in Declination of the Earth's Magnetic Field at Palo Alto, California, and Tuscon, Arizona, for 12 March 1955.

Vertical Scale: 1 cm = 5 minutes

Figure 17

CHAPTER VI

CONCLUSIONS

The results obtained with this system indicated that measurement sensitivity comparable to that obtained at a Magnetic Observatory can easily be achieved. It is significant that the records shown in Chapter V were obtained without compensation for temperature variations or allowance for the presence of nearby magnetic objects. The accuracy of this new type variometer is dependent mainly upon the precision with which the frequency difference between the receiver signals for the two bias conditions can be measured.

The advantages of this system are numerous. Minute perturbations of the earth's magnetic field magnitude or direction can be indicated immediately on a permanent record in a period of 30 seconds or less. Such a device can be made compact, rugged, and mobile. The precession signal can be easily telemetered from an unattended unit or allowed to make a permanent record for long periods of time. The system sensitivity can be controlled quite simply by means of regulating the bias field current.

A limitation of this technique arises from the fact that a free precession magnetometer is essentially an averaging device for it responds to the average magnitude of the magnetic field over the counting interval. Hence, it cannot be used to observe high frequency variations of the various components.

Suggested applications for this system would first include use in a station magnetometer as a primary standard.

This system might also hold promise as a navigational device. It is well known that many areas of the ocean contain magnetic anomalies with

sharp gradients. It is conceivable that positional accuracy comparable to that now obtained on surface ships by astronomical observations may result from a simple application of the technique by submarines below the surface of the ocean.

Further investigation may indicate an application of this system in harbor defense. The employment of two sensing units, separated by a predetermined distance in order to balance out the diurnal variations, would provide magnetic information by responding to variations in declination or inclination.

Operation of the programming circuit without energizing the bias field will provide a record of the gradient of the magnetic field at the sensing head. This application may be useful in the plotting of small fields.

The magnitude and direction of the earth's magnetic field may be quickly and easily determined in any location by this system. A single set of bias field coils mounted in a gimbal for precision adjustments would be sufficient for measuring both declination and inclination. A particular advantage of this instrument would be in the measurement of angles of inclination near the vertical. Since the mechanism of measurement does not depend upon the force of gravity, the accuracy of measurement would be unchanged in any orientation.

In the experimental equipment used a change of one milliampere in the bias field coils caused a change in F of two gammas. This suggests a possible application for this technique for rapid and accurate calibration of ammeters.

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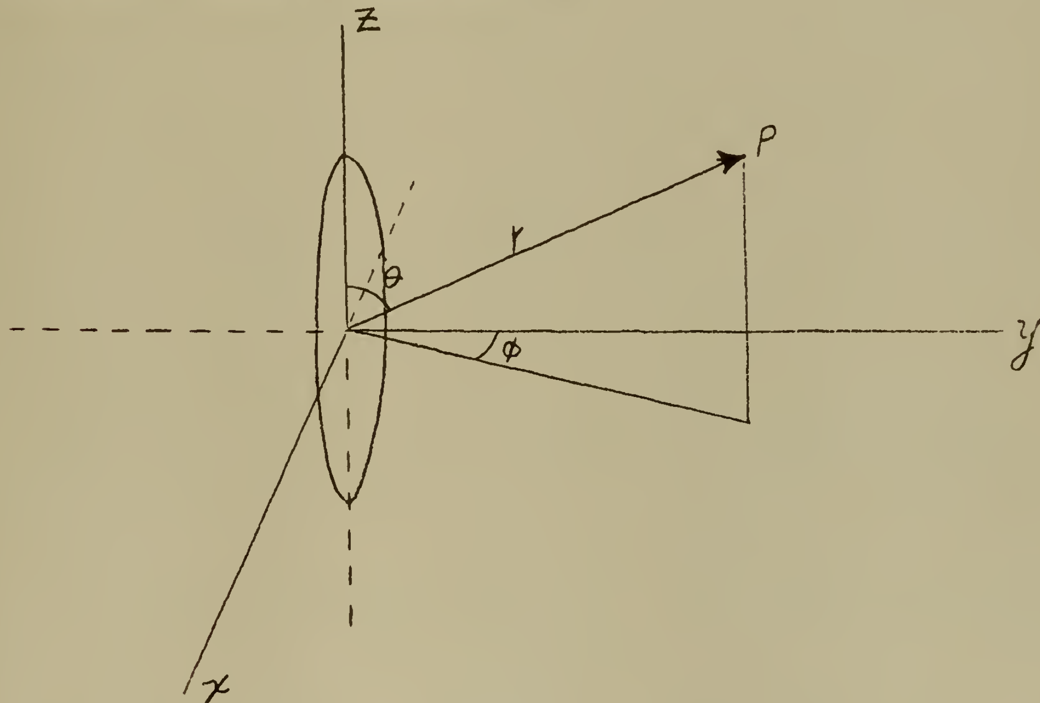
APPENDIX I

MAGNETIC FIELD EXPRESSIONS FOR ONE, TWO, AND THREE

COIL COMBINATIONS

1. Single coil.

The general expression for the field of a circular current loop is derived in Smythe [12] section 7.06. A spherical coordinate system is used as shown in the following diagram:



When the origin is taken at the center of the loop, the field may be expressed:

$$B_r = \frac{-2\pi\mu I}{a} \sum_{n\ \text{odd}}^{1\to\infty} (-1)^{\frac{n+1}{2}} \frac{(1\cdot 3\cdots\cdots n)\cdot(n+1)}{(2\cdot 4\cdots)(n-1)(n+1)} \left(\frac{r}{a}\right)^{n-1} P_n(\cos\theta)$$

$$B_\theta = \frac{2\pi\mu I}{a} \sum_{n\ \text{odd}}^{1\to\infty} (-1)^{\frac{n+1}{2}} \frac{(1\cdot 3\cdots\cdots n)\cdot(n+1)}{n\cdot(2\cdot 4\cdots\cdots)(n-1)(n+1)} \left(\frac{r}{a}\right)^{n-1} P_n'(\cos\theta)$$

In order to obtain an expression for field uniformity, consider the B_r term and write the first few terms of the series:

$$B_r = \frac{2\pi\mu I}{a}\left[P_1(\cos\theta) - \frac{3}{2}\left(\frac{r}{a}\right)^2 P_3(\cos\theta) + \frac{15}{8}\left(\frac{r}{a}\right)^4 P_5(\cos\theta) + \dots \right]$$

It is seen that the first term of this series is a constant term. The next term varies as the square of the distance from the origin and since (r/a) will always be less than unity this will be the dominant error term. Therefore, the ratio of the constant field term B_r to the error field term ΔB_r is

$$\frac{B_r}{\Delta B_r} = \frac{P_1(\cos\theta)}{(-3/2)\left(\frac{r}{a}\right)^2 \; P_3(\cos\theta)} \qquad \text{for } r < a$$

2. Two coil case.

In the two coil case it is convenient to express the field for each coil of radius b as:

$$B_r = \frac{2\pi\mu I \sin\alpha}{b} \sum_{n=1}^{\infty} \left(\frac{r}{b}\right)^{n-1} P_n'(\cos\alpha) \; P_n(\cos\theta)$$

$$\text{for } r < b$$

$$B_\theta = \frac{-2\pi\mu I \sin\alpha}{b} \sum \frac{1}{n}\left(\frac{r}{b}\right)^{n-1} P_n'(\cos\alpha) \; P_n'(\cos\theta)$$

$$\text{for } r < b$$

Now, taking the origin at the center of the axis of two coaxial current loops, the total radial field at any point within the sphere $r < b$ may be written:

$$B_r = \frac{4\pi\mu I}{b}\Big[\sin\alpha\, P_1'(\cos\alpha)\, P_1(\cos\theta)$$
$$+ \sin\alpha \left(\frac{r}{b}\right)^2 P_3'(\cos\alpha)\, P_3(\cos\theta)$$
$$+ \sin\alpha \left(\frac{r}{b}\right)^4 P_5'(\cos\alpha)\, P_5(\cos\theta)$$
$$+ \;\ldots\ldots\Big]$$

In order to minimize the error field, choose the term $\cos\alpha$ so that $P_3'(\cos\alpha)$ is zero. This will occur for $\cos\alpha = \sqrt{5}$. Hence, the coils should be separated a radius apart, the Helmholtz condition.

The ratio of the constant term B_r to the dominant error term ΔB_r for this condition will be:

$$\frac{B_r}{\Delta B_r} = \frac{P_1'(\cos\alpha)\, P_1(\cos\theta)}{\left(\frac{r}{b}\right)^4 P_5'(\cos\alpha)\, P_5(\cos\theta)}$$

for $r < b$.

3. Three coil case.

The three coil case is a superposition of the two foregoing situations in which the origin will be taken at the center of a current loop of radius a and two coaxial current loops of radius b placed at equal distance on either side. Again, for simplicity, only the B_r expression will be considered.

Combining the expressions for the total radial field at any point, we have:

$$
\begin{aligned}
B_r = 2\pi\mu I\Big[& -\frac{1}{a}(-1)\left(\frac{r}{a}\right)^0 P_1(\cos\theta) + \frac{2}{b}\sin\alpha\, P_1'(\cos\alpha)\, P_1(\cos\theta) \\
& -\frac{1}{a}(-1)^2\left(\frac{3}{2}\right)\left(\frac{r}{a}\right)^2 P_3(\cos\theta) + \frac{2}{b}\sin\alpha\left(\frac{r}{b}\right)^2 P_3'(\cos\alpha)\, P_3(\cos\theta) \\
& -\frac{1}{a}(-1)^3\left(\frac{15}{8}\right)\left(\frac{r}{a}\right)^4 P_5(\cos\theta) + \frac{2}{b}\sin\alpha\left(\frac{r}{b}\right)^4 P_5'(\cos\alpha)\, P_5(\cos\theta) \\
& -\frac{1}{a}(-1)^4\left(\frac{35}{16}\right)\left(\frac{r}{a}\right)^6 P_7(\cos\theta) + \frac{2}{b}\sin\alpha\left(\frac{r}{b}\right)^6 P_7'(\cos\alpha)\, P_7(\cos\theta) \\
& + \cdots\cdots\Big].
\end{aligned}
$$

Rearranging in terms of the coefficients of P_n $(\cos\theta)$,

$$
\begin{aligned}
B_r = 2\pi\mu I\Big\{ & \left[\frac{1}{a} + \frac{2}{b}\sin\alpha\, P_1'(\cos\alpha)\right] P_1(\cos\theta) \\
& + \left[-\frac{3}{2a}\left(\frac{r}{a}\right)^2 + \frac{2}{b}\sin\alpha\left(\frac{r}{b}\right)^2 P_3'(\cos\alpha)\right] P_3(\cos\theta) \\
& + \left[\frac{15}{8a}\left(\frac{r}{a}\right)^4 + \frac{2}{b}\sin\alpha\left(\frac{r}{b}\right)^4 P_5'(\cos\alpha)\right] P_5(\cos\theta) \\
& + \left[-\frac{35}{16a}\left(\frac{r}{a}\right)^6 + \frac{2}{b}\sin\alpha\left(\frac{r}{b}\right)^6 P_7'(\cos\alpha)\right] P_7(\cos\theta) \\
& + \cdots\cdots\Big\}.
\end{aligned}
$$

Next set the coefficient of P_3 $(\cos\theta)$ equal to zero:

$$-\frac{3}{2a}\left(\frac{r}{a}\right)^2 + \frac{2}{b}\sin\alpha\left(\frac{r}{b}\right)^2 P_3'(\cos\alpha) = 0$$

or

$$\sin\alpha\; P_3'(\cos\alpha) = \frac{3}{4}\left(\frac{b}{a}\right)^3$$

For convenience let

$$
\begin{aligned}
\cos\alpha &= x \\
\sin\alpha &= \sqrt{1-x^2}
\end{aligned}
$$

Then

$$P_3'(x) = \frac{3}{2}\sqrt{1-x^2}\,(5x^2-1)$$

Substituting,

$$\frac{3}{2}(1-x^2)(5x^2-1) = \frac{3}{4}\left(\frac{b}{a}\right)^3$$

$$\frac{b}{a} = \left[2(1-x^2)(5x^2-1)\right]^{1/3}$$

Similarly, set the coefficient of P_5 (cos θ) equal to zero.

$$\frac{15}{8a}\left(\frac{r}{a}\right)^4 + \frac{2}{b}\sin\alpha\left(\frac{r}{b}\right)^4 P_5'(\cos\alpha) = 0$$

Which reduces to

$$\sin\alpha \; P_5'(\cos\alpha) = -\frac{15}{16}\left(\frac{b}{a}\right)^5$$

or, since

$$P_5'(x) = \frac{105}{8}\sqrt{1-x^2}\left[3x^4-2x^2+\frac{1}{7}\right]$$

$$\frac{105}{8}(1-x^2)(3x^4-2x^2+\frac{1}{7}) = -\frac{15}{16}\left(\frac{b}{a}\right)^5$$

$$\left(\frac{b}{a}\right)^5 = 14(x^2-1)(3x^4-2x^2+\frac{1}{7})$$

Substituting for (b/a),

$$\left[2(1-x^2)(5x^2-1)\right]^5 = \left[14(1-x^2)(2x^2-3x^4-\frac{1}{7})\right]^3$$

This equation was solved graphically for the root between zero and plus one with the result:

$$x^2 = 0.41 = \cos^2\alpha$$

$$\alpha = 50.2^\circ$$

$$\left(\frac{b}{a}\right) = \left[2(1-.41)(5(.41)-1)\right]^{1/3}$$

$$\frac{b}{a} = 1.074$$

Under these conditions the dominant error term will vary as the sixth power of the distance from the center of the coordinant system. Hence, the ratio of the constant term B_r to the error term ΔB_r will be:

$$\frac{B_r}{\Delta B_r} = \frac{(2.1)\, P_1(\cos\theta)\, a^7}{(-4.97)\, P_7^0(\cos\theta)\, a\, r^6}$$